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TECHNICAL REPORT

WHITE OAK LABORATORY

METHODS FOR SOLVING THE VISCOELASTICITY EQUATIONS FOR CYLINDER SPHERE PROBLEMS

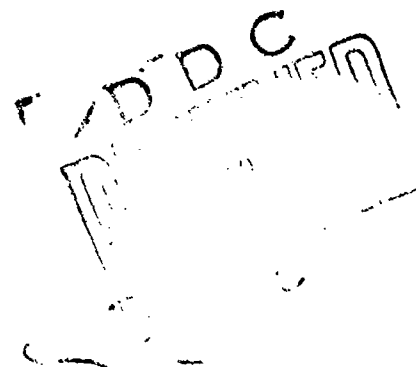
BY
G.C. Gaunard

22 MARCH 1976

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This report considers techniques used to solve the Navier field equations of viscoelasticity in the Kelvin-Voigt or Maxwell models, for cylindrical or spherical geometries. Introducing scalar and vector potentials into the viscoelasticity equations formulation, ultimately yields "telegraph-type" partial differential equations governing those potentials. For harmonic time-dependence, these reduce to scalar and vector Helmholtz's equations with complex propagation constants. These constants are shown to be related to the viscoelastic material			

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constants in a more or less complicated fashion depending on the viscoelastic model used. The stresses, strains and displacements are then found from these potentials for a dozen cases of interest in those two coordinate systems. The formulation resembles that of electrodynamics in a Coulomb gauge.)

The above information is vital to set-up and solve various kinds of boundary-value-problems of dynamic viscoelasticity which appear when studying cases of acoustic scattering from sound-absorbing structures. Problems we are now addressing. The analysis is summarized in two large Tables set-up in a conveniently accessible form. Remarks on "complex-moduli" are examined in a final section under the light of the viscoelasticity "Correspondence Theorem" and a list of recommendations and conclusions is given at the end.

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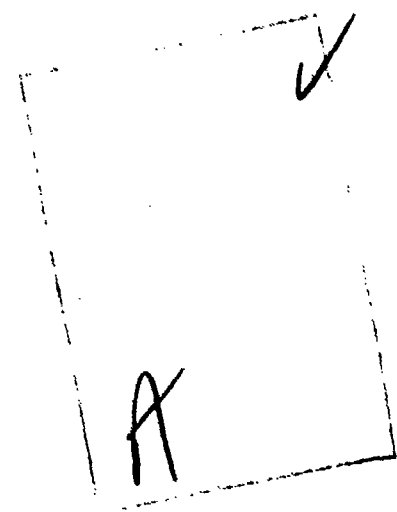
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METHODS FOR SOLVING THE VISCOELASTICITY EQUATIONS FOR CYLINDER
AND SPHERE PROBLEMS

This report describes techniques used to solve the field equations of dynamic viscoelasticity in two geometries, two viscoelastic models, and a dozen different situations of interest. This information is needed to study acoustic scattering from sound-absorbing structures of cylindrical and spherical shapes, which are cases presently under study by the author. This report sets up most of the theoretical viscoelasticity foundations needed to deal with the other sound scattering problems under study.

This work is continuing and it was done as part of an NSWC project entitled "Acoustical Properties of Ordnance Materials", Task No. MAT-03L-000/ZR00-001-010, Problem 127, which deals with acoustic scattering from objects covered with viscoelastic materials. This is a progress report describing work done during FY 76.

J. R. DIXON
By direction



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I. INTRODUCTION The basic viscoelastic models.

The deformation of a viscoelastic solid under any kind of external loads is usually studied by means of viscoelastic models. Two basic models commonly used are associated with the names of Kelvin¹-Voigt² and Maxwell³. In the Kelvin-Voigt model, the elastic and viscous properties of each material point (or particle) of the body, which can be respectively represented by a spring and a dashpot, are assumed connected in parallel. In the Maxwell model they are assumed connected in series. We repeat that this description applies at each point in the viscoelastic solid and it is as if the body contained a continuous distribution of damped oscillators, (viz, elementary mass-spring-dashpot systems) one at each one of its material points. As if it were not clear enough already, the entire "Kelvin-Voigt solid" can not be replaced by ONE spring connected in parallel with ONE dashpot. This simplistic view of a deformable solid as a single particle, may be useful in some other elementary context such as that used when one treats a body as a particle, but it is of no use in viscoelasticity. Otherwise viscoelasticity could not be viewed as a field-theory capable of describing stress and displacement fields at each point in a body, since by that oversimplification, the body has been reduced to a particle. We emphasize this rather trivial, but quickly forgotten point, because it is common to find authors who try to use, say, the "Kelvin model" as a single spring in parallel with a single dashpot, only to find that the model is "no-good" and that they must go to "more-degrees-of-freedom systems", such as three "Kelvin-models" in series, or other similar configurations, to obtain meaningful results. It is clear that this approach does not give the Kelvin model, as it truly is, even a chance to "work". These authors have replaced the continuous field-equations of viscoelasticity⁴, by a set of three ordinary second-order differential equations of the simple

¹ William Thomson (Lord Kelvin), b. at Belfast, 1827; d. near Glasgow, 1907. Professor at Glasgow Univ, 1846-1889. Buried at Westminster Abbey, London (near Newton's tomb).

² Woldemar Voigt, b. at Leipzig 1850; d. at Göttingen 1919. Professor of Mechanics at Göttingen Univ, Germany.

³ James C. Maxwell, b. Edinburgh, 1831; d. Cambridge, 1879. Professor of Physics at King's College and at Cambridge Univ, 1860-1879. Founder of the Cavendish Lab at Cambridge.

⁴ ie, a set of three scalar partial differential equations hopelessly coupled and non-linear, governing the displacement-field in the body, which can be linearized for small deformations and small deformation-gradients, and the "linear" theory of viscoelasticity then results. These equations are called the Navier equations of viscoelasticity.

damped-oscillator type. This replacement is, in no way, equivalent to solving the field-equations of Navier.

The basic point of the continuum field-theory approach to viscoelasticity is that the model assumptions, that the spring and dashpot are connected in series or in parallel at each point in the body, are immediately reflected in the fact that the field-equations resulting from either one of those models turn out to be different. To fix the ideas we now give the linearized form of the field-equations for both these models.

i) Kelvin-Voigt Model (parallel):

$$\left[1 + \frac{\mu_v}{\mu_e} \frac{\partial}{\partial t}\right] \nabla^2 \vec{u} + \left(\frac{\lambda_e + \mu_e}{\mu_e}\right) \left[1 + \frac{\lambda_v + \mu_v}{\lambda_e + \mu_e} \frac{\partial}{\partial t}\right] \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) = \frac{1}{C_s^2} \frac{\partial^2 \vec{u}}{\partial t^2}$$

ii) Maxwell Model (series):

$$\left[1 + \frac{1}{2\beta} \frac{\partial}{\partial t}\right] \left[\nabla^2 \vec{u} + \left(\frac{\lambda_e + \mu_e}{\mu_e}\right) \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \frac{1}{C_s^2} \frac{\partial^2 \vec{u}}{\partial t^2}\right] + \frac{3\alpha}{2\beta} \left[\nabla^2 \vec{u} + \frac{1}{3} \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \frac{1}{C_s^2} \frac{\partial^2 \vec{u}}{\partial t^2}\right] = \frac{1}{C_s^2} \frac{\partial^2 \vec{u}}{\partial t^2} + \frac{3\alpha + 2\beta}{C_s^2} \frac{\partial \vec{u}}{\partial t} \quad (C_s^2 = \frac{\mu_e}{\rho})$$

Here ρ is the material density, λ , μ are the elastic Lamé constants, \vec{u} is the vector-displacement field, and λ_v, μ_v (or α, β in the Maxwell model) are the viscosity coefficients. As given above, in differential operator form, these equations hold in any coordinate system. These are the Navier equations one must solve in the viscoelastic body. It is possible to derive eqs. (i) and (ii) starting from the basic idea that the spring and dashpot at each material point are connected either in parallel or in series respectively. Once the displacement field components are found by solving (i) or (ii) with suitable boundary conditions, one can then find the stresses from them.

In the absence of viscosity (ie, $\lambda_v = 0, \mu_v = 0$ for the Kelvin model or $\alpha = 0, \beta = 0$ for the Maxwell model) both field equations (i) and (ii) given above reduce to,

iii) The Field equations of linear dynamic elasticity:

$$\nabla^2 \vec{u} + \left(\frac{\lambda_e + \mu_e}{\mu_e}\right) \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) = \frac{1}{C_s^2} \frac{\partial^2 \vec{u}}{\partial t^2}$$

since now there are only springs and no dashpots at each material point. The scalar components of this vector equation are the Navier equations of elasticity in the absence of body-forces as found by Navier⁵ for the elastic body. Equations (i) and (ii) are also called the Navier equations (of viscoelasticity) by extension, since Navier never worked with viscoelastic solids.

In most materials of interest, the "constants" λ_e , μ_e and the "coefficients" λ_v , μ_v are really not constants, but frequency-dependent parameters. Thus, few materials are completely describable by the Kelvin or the Maxwell models in the continuum sense of eqs. (i) and (ii). The material behavior of viscoelastic substances tends to follow one or the other model in different regions of the parameters involved, say, frequency among others. This means that in general, these models of field-equations (i) and (ii) are quite "good". Rubbers at low frequencies are known to be well described by the Kelvin model (i). Pulse-tube measurements exploit this factual observation. Since these models are not perfect, researchers in this field have proposed more complicated models.

It is not hard to see that various Maxwell elements in series at each point in the body have the properties of a single Maxwell element with equivalent spring

and dashpot constants given by $1/k_{eq} = \sum_{i=1}^n 1/k_i$ and $1/\eta_{eq} = \sum_{i=1}^n 1/\eta_i$

respectively. Various Kelvin elements in parallel at each point in the body have

the properties of a single Kelvin element with $k_{eq} = \sum_{i=1}^n k_i$ and $\eta_{eq} = \sum_{i=1}^n \eta_i$.

On the other hand, Kelvin elements in series, or Maxwell elements in parallel, have more complicated properties. In their desire to generalize the basic models (i) and (ii), researchers have invented the so called "standard viscoelastic model". It consists of a Maxwell element in parallel with a Kelvin element at each point in the body. The linearized field equations which result in this model when two springs and two dashpots are connected as described above must be very complicated and rare, since I can not find one single reference to them. I was able to derive the particular subcase which results when the two dashpots are described by one single viscous constant η and both springs by the same two elastic constants λ_e and μ_e . The resulting field equations in this still very

⁵ C. L. Navier, (1827) Mémoire sur les lois de l'équilibre et du mouvement des corps solides élastiques. Mém. Acad. Sci. Paris 7.

general case are,

$$\begin{aligned}
 \text{iv) } \mu_e \left[1 - \delta_{t_1 t_2} \right] \nabla^2 \vec{u} + \left[\lambda_e + \mu_e - \frac{\mu_e}{3} \delta_{t_1 t_2} \right] \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) + \\
 + t_1 \frac{\partial}{\partial t} \left\{ (\lambda_e + \frac{2}{3} \mu_e) \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \rho \vec{u} \right\} + \mu_e t_2 \frac{\partial}{\partial t} \left\{ \nabla^2 \vec{u} + \frac{1}{3} \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) \right\} = \\
 = \rho \vec{u}
 \end{aligned}$$

where $\delta_{t_1 t_2}$ is the Kronecker delta equal to one (zero) for $t_1 = t_2$ (or $t_1 \neq t_2$).

Further t_1 (or t_2) equal η/μ_e . For $t_1 = 0$ and $t_2 = \eta/\mu_e$ (here $\eta = \mu_v$), eqs. (iv) reduce to the Kelvin model equations (i). When $t_1 = t_2 = \eta/\mu_e$ (which amounts to setting $2\beta = 1/\eta$ and $-3\alpha = 1/\eta$), equations (iv) reduce to the Maxwell model equations (ii) with $3\alpha + 2\beta = 0$. The quantities t_1 and t_2 are the retardation or relaxation times of the Kelvin and Maxwell models respectively. The constitutive (ie, stress-strain) relations of this viscoelastic model are also given in ref (6). I know of no viscoelastic boundary-value-problem that has ever been analytically solved using this model, which many agree is more realistic than (i) or (ii), since it can describe wider material behaviors and a wider variety of materials. Since the "standard viscoelastic model" is so hard to handle, we must realistically conclude that all analytical viscoelastic problems we are bound to see solved in the near future will be based on either the Kelvin or the Maxwell models of field equations (i) or (ii) respectively.

Occasionally we find a reference in the literature which contains a very complicated network such as a dozen "Maxwell elements" in parallel. Such a complicated network immediately implies that this is not a continuum field-theory approach, but rather that the body has been replaced by twelve coupled damped-oscillators. Work of this nature happens to be mostly experimental, chemical, and containing little mathematical analysis. These complicated networks are basically intended as pictorial descriptions without much physico-mathematical discussion of the response of the twelve coupled-oscillators, which, per se, is far from being a trivial problem. Sometimes one comes across an entire textbook dealing with various aspects of viscoelasticity without a single reference to the continuum approach, or the field-equations for the viscoelastic models. This old fashioned tendency is out-dated today. Mechanics of deformable media has become more highly mathematized now-a-days, than ever before.

II. SOLUTION OF THE FIELD EQUATIONS. (Tables 1 & 2)

We will now present a way to solve the field-equations of linear dynamic viscoelasticity (i) and (ii). The technique we will follow is common in other field-theories (ie, electro-dynamics) but we believe it is novel in viscoelasticity. It consists of introducing scalar and vector potentials ϕ and $\vec{\psi}$ such that $\vec{u} = \vec{\nabla}\phi + \vec{\nabla} \times \vec{\psi}$ and $\vec{\nabla} \cdot \vec{\psi} = 0$. By splitting the displacement field \vec{u} into irrotational and solenoidal parts in this fashion, it turns out that the field equations of each model (i) or (ii) are automatically satisfied provided that the scalar and vector potentials satisfy certain scalar and vector telegraph-type equations. The vector potential has three scalar components, but since the solenoidal gauge condition $\vec{\nabla} \cdot \vec{\psi} = 0$ must be satisfied, only two of the three scalar functions are independent. Those two (ψ and χ where ψ is not $|\vec{\psi}|$) together with the scalar potential ϕ , form the three independent scalar potentials that can be used to solve the problem. If these three independent scalar potentials satisfy three scalar telegraph-type equations then it can be shown that the field equations are automatically satisfied.

It is then possible to express all the stress and displacement components in terms of these independent potentials, which are determined first by solving the telegraph-type equations they must satisfy. In this fashion we can determine all the stresses and displacements in the body needed to completely solve the problem.

To discuss these solutions for the Kelvin-Voigt or Maxwell solids we have constructed two charts. (Tables 1 and 2). Table 1 deals with these viscoelastic models for the general case of arbitrary time-dependence. Table 2 covers the important case of harmonic time-dependence of the form $\exp(-i\omega t)$. The coordinate systems covered in those tables are the cylindrical and the spherical. The cylindrical system is studied in general in columns B and C for the Kelvin or Maxwell models respectively. It is also studied in the (3-independent) plane-strain subcase, which is applicable to infinitely long cylinders, in columns D and E, for the Kelvin and Maxwell models, respectively. The spherical system is also presented in full generality in column F for the Kelvin model. It is also given in column G for the axi-symmetric case without azimuthal dependence ϕ , again for the Kelvin model. The Maxwell model is not covered for spherical coordinates in these tables. The basic result of these tables, particularly Table 2, for harmonic time-dependence, is that if we introduce independent scalar potentials which satisfy the Helmholtz's equations with complex propagation constants given in item (10), then the field equations given in items (7) are automatically satisfied for each of the cases considered. Furthermore, the displacement and stress-field components are found from the potentials by the relations in items (4) and (8) respectively. Notice that the complex propagation constants given in item (10) are related to the elastic and viscous constants of the viscoelastic material

TABLE 1: EQUATIONS OF LINEAR DYNAMIC VISCOELASTICITY IN CYLINDRICAL AND SPHERICAL COORDINATES

A) GENERAL FORMULAS VALID FOR ANY VISCOELASTIC MODEL IN ANY COORDINATE SYSTEM.	A) KELVIN-VOYSEKOVSKAYA MODEL (PARALLEL) FOR GENERAL 5-DIMENSIONAL CYLINDRICAL COORDINATES.
<p>1) RELATION BETWEEN THE TRACES ϵ_{AA} AND ϵ_{RR}:</p> $\epsilon_{AA} = \left(\frac{3P+2Q}{3R+2S} \right) \epsilon_{RR}$ <p>ϵ_{AA} - HYDROSTATIC OR BULK STRESS, ϵ_{RR} - DILATATION.</p>	<p>1) RELATION BETWEEN THE TRACES, ϵ_{AA} AND ϵ_{RR}:</p> $\epsilon_{AA} = [(3\lambda_0 + 2\mu_0) + (3\lambda_v + 2\mu_v) \frac{\partial}{\partial t}] \epsilon_{RR}$ <p>λ_0, μ_0 - ELASTIC (LAKE) CONSTANTS, λ_v, μ_v - VISCOUS CONSTANTS</p>
<p>2) CONSTITUTIVE RELATIONS:</p> $R \delta_{ij} \epsilon_{AA} + 2S \epsilon_{ij} = P \delta_{ij} \epsilon_{AA} + 2Q \epsilon_{ij} \quad \text{OR, WHICH IS THE SAME,}$ $\epsilon_{ij} = \frac{2(PS - RQ)}{2S(3R + 2S)} \delta_{ij} \epsilon_{AA} + \frac{2Q}{2S} \epsilon_{ij}, \quad \text{WHERE, } \epsilon_{AA} = \Delta = \nabla \cdot \vec{u}$ <p>$R = \alpha + \sum_{k=1}^n \alpha_k \frac{\partial^k}{\partial t^k}, S = \beta + \sum_{k=1}^n \beta_k \frac{\partial^k}{\partial t^k}, P = \lambda + \sum_{k=1}^n \lambda_k \frac{\partial^k}{\partial t^k}, Q = \mu + \sum_{k=1}^n \mu_k \frac{\partial^k}{\partial t^k}$</p>	<p>2) CONSTITUTIVE RELATIONS:</p> <p>HERE, $R = 0, S = \frac{1}{2}, P = \lambda_0 + \lambda_v \frac{\partial}{\partial t}, Q = \mu_0 + \mu_v \frac{\partial}{\partial t}$</p> $\epsilon_{ij} = [\lambda_0 + \lambda_v \frac{\partial}{\partial t}] \delta_{ij} \epsilon_{AA} + 2(\mu_0 + \mu_v \frac{\partial}{\partial t}) \epsilon_{ij}$ <p>CLEARLY, $\alpha = 0, \alpha_k = 0 \quad \forall k = 1, 2, \dots, n, \beta = \frac{1}{2}, \beta_k = 0 \quad \forall k = 1, 2, \dots, n, \lambda = \lambda_0, \mu = \mu_0, \lambda_k = \lambda_v, \mu_k = \mu_v$</p>
<p>3) STRAIN-DISPLACEMENT RELATIONS:</p> $\epsilon_{ij} = u_{(i,j)} = \frac{1}{2} [u_{i,j} + u_{j,i}]$	<p>3) STRAIN-DISPLACEMENT RELATIONS:</p> $\epsilon_{AA} = \frac{\partial u_r}{\partial r}, \epsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{u_r}{r}, \epsilon_{zz} = \frac{\partial u_z}{\partial z}, 2\epsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$
<p>4) DISPLACEMENT-INDEPENDENT POTENTIAL RELATIONS:</p> <p>NO GENERAL EXPRESSION CAN BE GIVEN VALID IN ANY COORD. SYSTEM.</p>	<p>4) DISPLACEMENT-INDEPENDENT POTENTIAL RELATIONS:</p> <p>$\vec{u} = \vec{\nabla}(\varphi + \frac{\partial \psi}{\partial t}) - \vec{e}_r \nabla^2 \varphi + \vec{e}_z \nabla^2 \psi = \vec{\nabla} \varphi + \vec{\nabla} \times (\vec{\nabla} \psi \times \vec{e}_z)$</p> <p>OR IN COMPONENT FORM:</p> $u_r = \frac{\partial}{\partial r} [\varphi + \frac{\partial \psi}{\partial t}] - \frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \psi), \quad u_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} (\varphi + \frac{\partial \psi}{\partial t}) + \frac{\partial}{\partial z} \psi$ <p>HERE: $\varphi = \varphi(r, \theta, z, t), \psi = \psi(r, \theta, z, t), \lambda = \lambda(r, \theta, z, t), \nabla = \nabla(r, \theta, z, t)$</p> <p>NOTE: $\vec{u} = \vec{\nabla} \varphi + \vec{\nabla} \times \vec{\psi}, \vec{\nabla} \cdot \vec{\psi} = 0$</p>
<p>5) STRESS-DISPLACEMENT RELATIONS:</p> $\epsilon_{ij} = \frac{1}{2S} \left(\frac{PS - RQ}{3R + 2S} \right) \delta_{ij} \Delta + \frac{2Q}{2S} \epsilon_{ij}, \quad \text{WHERE } \Delta = u_{AA} = \nabla \cdot \vec{u}$ $\epsilon_{ij} = \frac{1}{2} [u_{i,j} + u_{j,i}] \quad \text{AS IN (3).}$	<p>5) STRESS-DISPLACEMENT RELATIONS:</p> $\epsilon_{ij} = [\lambda_0 + \lambda_v \frac{\partial}{\partial t}] \delta_{ij} \Delta + 2(\mu_0 + \mu_v \frac{\partial}{\partial t}) \epsilon_{ij} \quad \text{WHERE}$ <p>ϵ_{ij} ARE GIVEN IN (3) AND $\Delta = \nabla \cdot \vec{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}$</p>
<p>6) STRAIN-INDEPENDENT POTENTIAL RELATIONS:</p> <p>NO GENERAL EXPRESSION CAN BE GIVEN VALID IN ANY COORD. SYSTEM.</p>	<p>6) STRAIN-INDEPENDENT POTENTIAL RELATIONS:</p> $\epsilon_{rr} = \frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} (\nabla^2 \psi)$ $\epsilon_{\theta\theta} = \left[\frac{1}{r} \frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) + \frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} (\nabla^2 \psi) \right] - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} (\nabla^2 \psi)$ $\epsilon_{zz} = \frac{\partial^2}{\partial z^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{\partial^2}{\partial z^2} (\nabla^2 \psi) = \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2}{\partial z^2} \left[\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} (\nabla^2 \psi) \right]$ $2\epsilon_{r\theta} = \frac{1}{r} \frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{1}{r} \frac{\partial^2}{\partial r^2} (\nabla^2 \psi) + \frac{\partial^2 \psi}{\partial r^2} (\nabla^2 \psi) = \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} (\nabla^2 \psi)$ $2\epsilon_{rz} = 2 \frac{\partial^2}{\partial r \partial z} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{\partial^2}{\partial r \partial z} (\nabla^2 \psi) = 2 \frac{\partial^2 \psi}{\partial r \partial z} + 2 \frac{\partial^2 \psi}{\partial r \partial z} \left(\frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} \right) + \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial^2}{\partial r^2} \right) (\nabla^2 \psi) = 2 \frac{\partial^2 \psi}{\partial r \partial z}$
<p>7) THE FIELD EQUATIONS: (FOR THE DISPLACEMENT FIELD)</p> <p>a):</p> $[QR + 2S(P + Q)] \nabla^2 (\nabla \cdot \vec{u}) + Q(3R + 2S) \nabla^2 \Delta = 2S(3R + 2S) p \vec{u}$ <p>b):</p> $2[2QR + S(P + 2Q)] \nabla^2 (\nabla \cdot \vec{u}) - Q(3R + 2S) \nabla^2 (\nabla \cdot \vec{u}) - 2S(3R + 2S) p \vec{u}$ <p>NOTE: $\nabla^2 \vec{u} = \nabla (\nabla \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u})$</p> <p>$Q, R, S, P$ ARE OPERATORS DEPENDING ON THE MODEL.</p>	<p>7) FIELD EQUATIONS:</p> <p>a): $[\mu_0 + \mu_v \frac{\partial}{\partial t}] \nabla^2 \vec{u} + [\lambda_0 + \lambda_v \frac{\partial}{\partial t} - (\lambda_0 + \lambda_v) \frac{\partial}{\partial t}] \nabla (\nabla \cdot \vec{u}) = p \frac{\partial^2 \vec{u}}{\partial t^2}$</p> <p>OR, IN COMPONENT FORM:</p> $[\mu_0 + \mu_v \frac{\partial}{\partial t}] [\nabla^2 u_r - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial \psi}{\partial r} \right) - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} (\nabla^2 \psi)] + [\lambda_0 + \lambda_v \frac{\partial}{\partial t} - (\lambda_0 + \lambda_v) \frac{\partial}{\partial t}] \frac{\partial}{\partial r} \left(\frac{\partial \psi}{\partial r} \right) = p \frac{\partial^2 u_r}{\partial t^2}$ $[\mu_0 + \mu_v \frac{\partial}{\partial t}] [\nabla^2 u_\theta - \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial \psi}{\partial r} \right) - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} (\nabla^2 \psi)] + [\lambda_0 + \lambda_v \frac{\partial}{\partial t} - (\lambda_0 + \lambda_v) \frac{\partial}{\partial t}] \frac{\partial}{\partial \theta} \left(\frac{\partial \psi}{\partial r} \right) = p \frac{\partial^2 u_\theta}{\partial t^2}$ $[\mu_0 + \mu_v \frac{\partial}{\partial t}] [\nabla^2 u_z] + [\lambda_0 + \lambda_v \frac{\partial}{\partial t} - (\lambda_0 + \lambda_v) \frac{\partial}{\partial t}] \frac{\partial}{\partial z} \left(\frac{\partial \psi}{\partial r} \right) = p \frac{\partial^2 u_z}{\partial t^2}$ <p>WHERE Δ AND ∇^2 ARE GIVEN ABOVE IN (4) AND (5).</p> <p>b): $[\lambda_0 + 2\mu_0 + (\lambda_0 + 2\mu_0) \frac{\partial}{\partial t}] \nabla (\nabla \cdot \vec{u}) - (\mu_0 + \mu_v \frac{\partial}{\partial t}) \nabla^2 (\nabla \cdot \vec{u}) = p \frac{\partial^2 \Delta}{\partial t^2}$</p> <p>THESE TWO FORMS (a) AND (b) ARE EQUIVALENT. ∇^2 GIVEN IN (6).</p>
<p>8) STRESS-INDEPENDENT POTENTIAL RELATIONS:</p> <p>NO GENERAL EXPRESSION CAN BE GIVEN VALID IN ANY COORD. SYSTEM.</p>	<p>8) STRESS-INDEPENDENT POTENTIAL RELATIONS:</p> $\epsilon_{rr} = [\lambda_0 + \lambda_v \frac{\partial}{\partial t}] \nabla^2 \varphi + 2(\mu_0 + \mu_v \frac{\partial}{\partial t}) \left[\frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} (\nabla^2 \psi) \right]$ $\epsilon_{\theta\theta} = [\lambda_0 + \lambda_v \frac{\partial}{\partial t}] \nabla^2 \varphi + 2(\mu_0 + \mu_v \frac{\partial}{\partial t}) \left[\frac{1}{r} \frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) + \frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} (\nabla^2 \psi) \right]$ $\epsilon_{zz} = [\lambda_0 + \lambda_v \frac{\partial}{\partial t}] \nabla^2 \varphi + 2(\mu_0 + \mu_v \frac{\partial}{\partial t}) \left[\frac{\partial^2}{\partial z^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{\partial^2}{\partial z^2} (\nabla^2 \psi) \right]$ $\epsilon_{r\theta} = [\mu_0 + \mu_v \frac{\partial}{\partial t}] \left[\frac{1}{r} \frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{1}{r} \frac{\partial^2}{\partial r^2} (\nabla^2 \psi) + \frac{\partial^2 \psi}{\partial r^2} (\nabla^2 \psi) \right]$ $\epsilon_{rz} = [\mu_0 + \mu_v \frac{\partial}{\partial t}] \left[2 \frac{\partial^2}{\partial r \partial z} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{\partial^2}{\partial r \partial z} (\nabla^2 \psi) + \frac{\partial^2 \psi}{\partial r \partial z} (\nabla^2 \psi) \right]$ <p>THESE ARE EQS. (5) WITH $\Delta = \nabla^2 \varphi$ AND ϵ_{ij} AS GIVEN IN EQS. (3)</p>
<p>9) TELEGRAPH-TYPE EQS. FOR SCALAR & VECTOR POTENTIALS:</p> <p>a): $[QR + 2S(P + Q)] \nabla^2 \varphi = 2S(3R + 2S) p \varphi$</p>	<p>9) TELEGRAPH-TYPE EQS. FOR THE SCALAR & VECTOR POTENTIALS:</p> <p>a): $[1 + M \frac{\partial}{\partial t}] \nabla^2 \varphi = \frac{1}{2} \frac{\partial^2 \varphi}{\partial t^2}, \quad M = \frac{\lambda_0 + \lambda_v}{\mu_0 + \mu_v}, \quad \varphi = \frac{p}{\mu_0 + \mu_v}$</p>

2 AND POLAR COORDINATES FOR THE NEUMANN PROBLEM IN THE PLANE AND IN VARIOUS SPECIAL CASES		
1) MODEL (PARALLEL) FOR CYLINDRICAL COORDINATES	2) MAXWELL-BOUGASSA FORM (SERIES) FOR GENERAL 3-DIMENSIONAL CYLINDRICAL COORDINATES	3) ELVIN JOINT VISCOELASTIC MODEL (PARALLEL) IN C DRICAL COORDS FOR THE PLANE STRAIN PROBLEM
1) RELATIONS $\epsilon_{\theta\theta}$ AND $\epsilon_{\phi\phi}$ $\mu_r \frac{\partial \epsilon}{\partial r} \in \epsilon_{\theta\theta}$ $\mu_r, \mu_\theta, \mu_\phi$ VISCOSUS CONSTANTS	1) RELATION BETWEEN THE STRAINS $\epsilon_{\theta\theta}$ AND $\epsilon_{\phi\phi}$ $\epsilon_{\theta\theta} = \frac{(3\lambda_0 + 2\mu_0) \frac{\partial \epsilon}{\partial r}}{3\alpha + 2\beta + \frac{\partial \epsilon}{\partial r}} \epsilon_{\phi\phi}$ $\epsilon_{\theta\theta}$ - BULK STRESS $\epsilon_{\phi\phi}$ - CUBICAL DILATATION - $\nabla \cdot \vec{u}$	1) DEFINITION OF THE PLANE STRAIN $\epsilon_{\theta\theta}$ THIS SITUATION, WHICH IS APPROPRIATE FOR FINITELY LONG THUS, HERE THERE IS NO ϕ -DEPENDENCE AND $u_r = u_r(r, \theta, t)$ HAVING $\phi = \phi(r, \theta, t)$, $\psi_r = 0$, $\psi_\theta = 0$ & $\psi_\phi = \psi_\phi(r, \theta, t)$ HAVING $\phi = \phi(r, \theta, t)$, $\psi_r = 0$ & $\psi_\theta = \psi_\theta(r, \theta, t)$ NOTE THAT THE FORMULATION IS INDEPENDENT OF ψ . ANY OF THESE
2) CONSTITUTIVE RELATIONS $\frac{\partial \epsilon}{\partial t} = 2 - \mu_0 - \mu_\theta \frac{\partial \epsilon}{\partial t}$ $2(\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t}) \epsilon_{\theta\theta}$ $\mu_0 = 1, 2, \dots, \mu_\theta = \frac{1}{2}$ $\mu_\phi = \mu_0, \lambda = \lambda_0, \mu = \mu_0$	2) CONSTITUTIVE RELATIONS HERE, $R = \alpha, S = \beta + \frac{1}{2} \frac{\partial \epsilon}{\partial t}, P = \lambda_0 \frac{\partial \epsilon}{\partial t}, Q = \mu_0 \frac{\partial \epsilon}{\partial t}$ $\epsilon_{ij} = \frac{\partial \epsilon}{\partial t} \left\{ \left[\frac{\lambda_0 (2\beta + \frac{\partial \epsilon}{\partial t}) - 2\alpha \mu_0}{3\alpha + 2\beta + \frac{\partial \epsilon}{\partial t}} \right] \delta_{ij} \epsilon_{\theta\theta} + 2\mu_0 \epsilon_{ij} \right\}$ CLEARLY, $\mu_0 = 0, \mu_1 = \mu_0, \mu_2 = 0 \forall k \neq 1, \lambda_0 = 0, \lambda_1 = \lambda_0, \lambda_2 = 0 \forall k \neq 1$ $\mu_0 \neq 0, \alpha_k = 0 \forall k = 1, 2, \dots, \beta \neq 0, \beta_1 = 0.5, \alpha_2 = 0 \forall k \neq 1$	2) CONSTITUTIVE RELATIONS $\begin{cases} \epsilon_{rr} = [\lambda_0 + \lambda_0 \frac{\partial \epsilon}{\partial t}] \Delta + 2(\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t}) \epsilon_{rr} \\ \epsilon_{\theta\theta} = [\lambda_0 + \lambda_0 \frac{\partial \epsilon}{\partial t}] \Delta + 2(\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t}) \epsilon_{\theta\theta} \\ \epsilon_{\phi\phi} = [\lambda_0 + \lambda_0 \frac{\partial \epsilon}{\partial t}] \Delta \end{cases} \quad \text{HERE } \Delta = \epsilon, \epsilon_{\theta\theta} = \frac{\partial \epsilon}{\partial t}$ $\epsilon_{r\theta} = [\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t}] 2 \epsilon_{r\theta}$
3) STRAIN DISPLACEMENT RELATIONS $\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \epsilon_{\phi\phi} = \frac{1}{r} \frac{\partial u_\phi}{\partial \phi}$	3) STRESS DISPLACEMENT RELATIONS $\epsilon_{ij} = \frac{\partial \epsilon}{\partial t} \left\{ \left[\frac{\lambda_0 (2\beta + \frac{\partial \epsilon}{\partial t}) - 2\alpha \mu_0}{3\alpha + 2\beta + \frac{\partial \epsilon}{\partial t}} \right] \delta_{ij} \Delta + 2\mu_0 \epsilon_{ij} \right\}$ WHERE, ϵ_{ij} ARE GIVEN IN (5) AND $\Delta = \frac{\partial \epsilon}{\partial t} + \frac{\mu_0}{r} + \frac{1}{r} \frac{\partial \mu_0}{\partial \theta} + \frac{\mu_\theta}{r} = \nabla \cdot \vec{u}$	3) STRESS DISPLACEMENT RELATIONS $\begin{cases} \epsilon_{rr} = [\lambda_0 + \lambda_0 \frac{\partial \epsilon}{\partial t}] \Delta + 2(\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t}) \epsilon_{rr} \\ \epsilon_{\theta\theta} = [\lambda_0 + \lambda_0 \frac{\partial \epsilon}{\partial t}] \Delta + 2(\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t}) \epsilon_{\theta\theta} \\ \epsilon_{\phi\phi} = [\lambda_0 + \lambda_0 \frac{\partial \epsilon}{\partial t}] \Delta \end{cases}$ WHERE, $\Delta = \frac{\partial \epsilon}{\partial t} + \frac{\mu_0}{r} + \frac{1}{r} \frac{\partial \mu_0}{\partial \theta} + \frac{\mu_\theta}{r}$
4) POTENTIAL RELATIONS $(\nabla^2 \phi) = \nabla^2 \phi - \nabla \cdot (\nabla \phi) \cdot \vec{e}_r + \nabla \cdot (\frac{\partial \phi}{\partial \theta} \vec{e}_\theta) - \vec{e}_r \cdot \nabla^2 \vec{r} = \nabla \cdot \nabla \phi + \nabla \cdot (\nabla \phi) \cdot \vec{e}_r + \nabla \cdot (\nabla \phi) \cdot \vec{e}_\theta$ $u_r = \frac{1}{2} \frac{\partial \phi}{\partial r} (\nabla^2 \phi) + \frac{\partial \phi}{\partial r} (\nabla^2 \phi)$ $u_\theta = \frac{1}{2} \frac{\partial \phi}{\partial \theta} (\nabla^2 \phi) - \nabla^2 \phi$ $u_\phi = \frac{1}{2} \frac{\partial \phi}{\partial \phi} (\nabla^2 \phi) - \nabla^2 \phi$ HERE, $\phi = \phi(r, \theta, t), \nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi}, \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$	4) STRESS INDEPENDENT POTENTIAL RELATIONS $\begin{cases} \epsilon_{rr} = \frac{\partial \phi}{\partial r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} (\nabla^2 \phi) \\ \epsilon_{\theta\theta} = \frac{1}{r} \frac{\partial \phi}{\partial \theta} - \frac{1}{r} \frac{\partial \phi}{\partial r} (\nabla^2 \phi) \\ \epsilon_{\phi\phi} = \frac{1}{r} \frac{\partial \phi}{\partial \phi} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} (\nabla^2 \phi) \end{cases}$ HERE, $\phi = \phi(r, \theta, t), \nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi}$	4) STRESS INDEPENDENT POTENTIAL RELATIONS $\begin{cases} \epsilon_{rr} = \frac{\partial \phi}{\partial r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} (\nabla^2 \phi) \\ \epsilon_{\theta\theta} = \frac{1}{r} \frac{\partial \phi}{\partial \theta} - \frac{1}{r} \frac{\partial \phi}{\partial r} (\nabla^2 \phi) \\ \epsilon_{\phi\phi} = \frac{1}{r} \frac{\partial \phi}{\partial \phi} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} (\nabla^2 \phi) \end{cases}$ THE OTHER STRAINS ($\epsilon_{r\theta}, \epsilon_{\theta r}, \epsilon_{\phi\theta}$) VANISH. IN THIS CASE, THE RESULT IS: $\epsilon_{r\theta} = \frac{1}{2} \frac{\partial \phi}{\partial r} - \frac{1}{2} \frac{\partial \phi}{\partial \theta} (\nabla^2 \phi), \quad \epsilon_{\theta r} = \frac{1}{2} \frac{\partial \phi}{\partial \theta} - \frac{1}{2} \frac{\partial \phi}{\partial r} (\nabla^2 \phi)$ THESE RELATIONS ARE FOUND BY SUBSTITUTING EQS.
5) FIELD EQUATIONS $\begin{aligned} & \left[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t} \right] \nabla \cdot (\nabla \phi) = \rho \frac{\partial \phi}{\partial t} \\ & \left[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t} \right] \frac{\partial \phi}{\partial r} = \rho \frac{\partial \phi}{\partial r} \\ & \left[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t} \right] \frac{\partial \phi}{\partial \theta} = \rho \frac{\partial \phi}{\partial \theta} \\ & \left[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t} \right] \frac{\partial \phi}{\partial \phi} = \rho \frac{\partial \phi}{\partial \phi} \end{aligned}$ ONE IN (4) AND (5). EQUVALENT, ∇^2 GIVEN IN (6).	5) FIELD EQUATIONS a) $\left[1 + \frac{1}{2} \frac{\partial \epsilon}{\partial t} \left(\frac{\lambda_0 + \mu_0}{\mu_0} \right) \nabla \cdot (\nabla \phi) - \frac{1}{2} \frac{\partial \phi}{\partial t} \right] = \frac{1}{2} \frac{\partial \phi}{\partial t} + \frac{1}{2} \frac{\partial \phi}{\partial t} \frac{\partial \epsilon}{\partial t}$ THIS CAN BE OPENED UP INTO COMPONENTS BY MEANS OF $\nabla^2 \phi = \nabla \cdot (\nabla \phi) \cdot \vec{e}_r + \nabla \cdot (\nabla \phi) \cdot \vec{e}_\theta + \nabla \cdot (\nabla \phi) \cdot \vec{e}_\phi$ $\nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \phi^2}$ b) $\left[1 + \frac{1}{2} \frac{\partial \epsilon}{\partial t} \left(\frac{\lambda_0 + \mu_0}{\mu_0} \right) \nabla \cdot (\nabla \phi) - \frac{1}{2} \frac{\partial \phi}{\partial t} \right] = \frac{1}{2} \frac{\partial \phi}{\partial t} + \frac{1}{2} \frac{\partial \phi}{\partial t} \frac{\partial \epsilon}{\partial t}$ THESE TWO FORMS (1) AND (2) ARE EQUIVALENT, ∇^2 GIVEN IN (6).	5) FIELD EQUATIONS a) $\left[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t} \right] \nabla \cdot (\nabla \phi) = \rho \frac{\partial \phi}{\partial t}$ b) $\left[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t} \right] \frac{\partial \phi}{\partial r} = \rho \frac{\partial \phi}{\partial r}$ c) $\left[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t} \right] \frac{\partial \phi}{\partial \theta} = \rho \frac{\partial \phi}{\partial \theta}$ d) $\left[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t} \right] \frac{\partial \phi}{\partial \phi} = \rho \frac{\partial \phi}{\partial \phi}$ WHERE, $\Delta = \frac{\partial \epsilon}{\partial t} + \frac{\mu_0}{r} + \frac{1}{r} \frac{\partial \mu_0}{\partial \theta} + \frac{\mu_\theta}{r}$ THE SECOND FORM (2) IS NOT AS USEFUL
6) FIELD EQUATIONS $\begin{aligned} & \left[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t} \right] \nabla \cdot (\nabla \phi) = \rho \frac{\partial \phi}{\partial t} \\ & \left[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t} \right] \frac{\partial \phi}{\partial r} = \rho \frac{\partial \phi}{\partial r} \\ & \left[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t} \right] \frac{\partial \phi}{\partial \theta} = \rho \frac{\partial \phi}{\partial \theta} \\ & \left[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t} \right] \frac{\partial \phi}{\partial \phi} = \rho \frac{\partial \phi}{\partial \phi} \end{aligned}$ ONE IN (4) AND (5). EQUVALENT, ∇^2 GIVEN IN (6).	6) STRESS-DEPENDENT POTENTIAL RELATIONS $\begin{aligned} \epsilon_{rr} &= \mu_0 \left[\lambda_0 \nabla^2 \phi + 2\mu_0 \left(\frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial r} + \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial \theta} \right) \right] + \mu_1 \\ \epsilon_{\theta\theta} &= \mu_0 \left[\lambda_0 \nabla^2 \phi + 2\mu_0 \left(\frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial r} + \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial \theta} \right) \right] + \mu_1 \\ \epsilon_{\phi\phi} &= \mu_0 \left[\lambda_0 \nabla^2 \phi + 2\mu_0 \left(\frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial r} + \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial \theta} \right) \right] + \mu_1 \\ \epsilon_{r\theta} &= \mu_0 \mu_\theta \left[\frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial r} \right] + \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \theta} (\nabla^2 \phi) \\ \epsilon_{\theta r} &= \mu_0 \mu_\theta \left[\frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial r} \right] + \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \theta} (\nabla^2 \phi) \\ \epsilon_{r\phi} &= \mu_0 \mu_\theta \left[2 \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \phi} \right] - \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \phi} (\nabla^2 \phi) \\ \epsilon_{\phi r} &= \mu_0 \mu_\theta \left[2 \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \phi} \right] - \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \phi} (\nabla^2 \phi) \end{aligned}$ WHERE, $\mu_1 = \frac{1}{2} \frac{\partial \epsilon}{\partial t} \left(\frac{\lambda_0 + \mu_0}{\mu_0} \right), \mu_2 = \frac{1}{2} \frac{\partial \epsilon}{\partial t} \left(\frac{\lambda_0 + \mu_0}{\mu_0} \right) - \frac{1}{2} \frac{\partial \epsilon}{\partial t} \left(\frac{\lambda_0 + \mu_0}{\mu_0} \right) \frac{\partial \epsilon}{\partial t}$ THESE ARE EQS. (5) WITH $\Delta = \nabla^2 \phi$ AND ϵ_{ij} AS GIVEN IN EQS. (4).	6) STRESS-DEPENDENT POTENTIAL RELATIONS $\begin{cases} \epsilon_{rr} = [\lambda_0 + \lambda_0 \frac{\partial \epsilon}{\partial t}] \nabla^2 \phi + 2[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t}] \left[\frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial r} + \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial \theta} \right] \\ \epsilon_{\theta\theta} = [\lambda_0 + \lambda_0 \frac{\partial \epsilon}{\partial t}] \nabla^2 \phi + 2[\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t}] \left[\frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial r} + \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial \theta} \right] \\ \epsilon_{\phi\phi} = [\lambda_0 + \lambda_0 \frac{\partial \epsilon}{\partial t}] \nabla^2 \phi \\ \epsilon_{r\theta} = [\mu_0 + \mu_\theta \frac{\partial \epsilon}{\partial t}] \left[2 \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \theta} \right] + \left(\frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial r} \right) \nabla^2 \phi \end{cases}$ TO HAVE THESE RELATIONS IN TERMS OF ϕ AND $\frac{\partial \phi}{\partial t}$ REPLACE $\nabla^2 \phi$ BY $\nabla^2 \phi$. ∇^2 IS GIVEN ABOVE IN THESE ARE EQS. (5) WITH $\Delta = \nabla^2 \phi$ AND ϵ_{ij} AS GIVEN IN EQS.
7) TELEGRAPH-TYPE EQS FOR THE SCALAR & VECTOR POTENTIALS	7) TELEGRAPH-TYPE EQS FOR THE SCALAR & VECTOR POTENTIALS	7) TELEGRAPH-TYPE EQS FOR THE SCALAR & VECTOR POTENTIALS

VI. VISCOELASTIC MODEL (PARALLEL) IN CYLINDRICAL COORDINATES FOR THE PLANE-STRAIN SUBCASE.

E) MAXWELL VISCOELASTIC MODEL (SERIES) IN CYLINDRICAL COORDINATES FOR THE PLANE-STRAIN SUBCASE.

F) KELVIN-VOIGT VISCOELASTIC MODEL (PARALLEL) IN CYLINDRICAL COORDINATES FOR THE PLANE-STRAIN SUBCASE.

OF THE PLANE-STRAIN CASE:

WHICH IS APPROPRIATE FOR INFINITELY LONG BODIES (i.e. CYLINDERS), OCCURS WHEN $u_z = 0$ AND $\frac{\partial}{\partial z}$ (ANY VARIABLE) = 0. THERE IS NO z -DEPENDENCE AND $u_r = u_r(r, \theta, t)$, $u_\theta = u_\theta(r, \theta, t)$. IN TERMS OF ϕ , ψ , χ , ψ , PLANE-STRAIN AMOUNTS TO (r, θ, t) , $\psi = 0$ & $\chi = \chi(r, \theta, t)$. OR IN TERMS OF THE INDEPENDENT POTENTIALS ϕ , ψ , χ , IT AMOUNTS TO (r, θ, t) , $\psi = 0$ & $\chi = \chi(r, \theta, t)$. NOTE THAT BY EQS. B.11-B, $\psi = -\nabla^2 \chi$. SO IN PLANE-STRAIN ONLY ϕ & χ APPEAR. DILATION IS INDEPENDENT OF ψ . ANY OF THESE EQUIVALENT DEFINITIONS YIELDS. $\epsilon_{zz} = 0$, $\epsilon_{rz} = 0$, & $\epsilon_{\theta z} = 0$.

1) CONSTITUTIVE RELATIONS:

$$\begin{aligned} \epsilon_{rr} &= \Delta + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{rr} \\ \epsilon_{\theta\theta} &= \Delta + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{\theta\theta} \\ \epsilon_{zz} &= 0 \end{aligned}$$

HERE $\Delta = \epsilon_{rr} + \epsilon_{\theta\theta} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}$

2) CONSTITUTIVE RELATIONS:

$$\begin{aligned} \epsilon_{rr} &= M_1 \{ \lambda_e \Delta + 2\mu_e \epsilon_{rr} \} + N_1 \\ \epsilon_{\theta\theta} &= M_1 \{ \lambda_e \Delta + 2\mu_e \epsilon_{\theta\theta} \} + N_1 \\ \epsilon_{zz} &= M_1 \{ \lambda_e \Delta \} + N_1 \\ \epsilon_{r\theta} &= M_1 \{ 2\mu_e \epsilon_{r\theta} \} \end{aligned}$$

(HERE M_1 AND N_1 ARE THE SAME OPERATORS GIVEN IN (C-8) OR E-8)

WHERE Δ IS GIVEN BELOW (i.e. $\epsilon_{rr} + \epsilon_{\theta\theta}$) IN (F).

3) STRAIN-DISPLACEMENT RELATIONS:

$$\epsilon_{rr} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \quad 2\epsilon_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \quad \epsilon_{zz} = 0, \quad 2\epsilon_{rz} = 0, \quad 2\epsilon_{\theta z} = 0$$

(EVERY STRAIN-COMPONENT WITH SUB-INDEX z , VANISHES.)

4) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$$\begin{aligned} \epsilon_{rr} &= \frac{1}{2} \nabla^2 \chi + \frac{1}{2} \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{2} \frac{\partial^2 \chi}{\partial \theta^2} \\ \epsilon_{\theta\theta} &= \frac{1}{2} \nabla^2 \chi + \frac{1}{2} \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{2} \frac{\partial^2 \chi}{\partial \theta^2} \\ \epsilon_{zz} &= 0 \end{aligned}$$

OR IN COMPONENT-FORM, IN TERMS OF ϕ AND χ . OR, IN TERMS OF ϕ AND ψ . ($\psi = -\nabla^2 \chi$)

5) STRESS-DISPLACEMENT RELATIONS:

$$\begin{aligned} \epsilon_{rr} &= \Delta + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{rr} \\ \epsilon_{\theta\theta} &= \Delta + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{\theta\theta} \\ \epsilon_{zz} &= 0 \end{aligned}$$

5) STRESS-DISPLACEMENT RELATIONS:

$$\begin{aligned} \epsilon_{rr} &= M_1 \{ \lambda_e \Delta + 2\mu_e \epsilon_{rr} \} + N_1 \\ \epsilon_{\theta\theta} &= M_1 \{ \lambda_e \Delta + 2\mu_e \epsilon_{\theta\theta} \} + N_1 \\ \epsilon_{zz} &= M_1 \{ \lambda_e \Delta \} + N_1 \\ \epsilon_{r\theta} &= M_1 \{ 2\mu_e \epsilon_{r\theta} \} \end{aligned}$$

(HERE M_1 & N_1 ARE THE SAME OPERATORS GIVEN IN (C-8) OR E-8)

6) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$$\epsilon_{rr} = \frac{1}{2} \nabla^2 \chi + \frac{1}{2} \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{2} \frac{\partial^2 \chi}{\partial \theta^2}, \quad 2\epsilon_{r\theta} = \frac{1}{2} \frac{\partial^2 \chi}{\partial r \partial \theta} + \frac{1}{2} \frac{\partial^2 \chi}{\partial \theta \partial r} - \frac{1}{2} \frac{\partial^2 \chi}{\partial r^2} - \frac{1}{2} \frac{\partial^2 \chi}{\partial \theta^2}$$

STRAINS (ϵ_{zz} , ϵ_{rz} , $\epsilon_{\theta z}$) VANISH. TO HAVE THESE RELATIONS IN TERMS OF ϕ AND ψ , WE REPLACE $\nabla^2 \chi$ BY $-\psi$. THE RESULT IS,

$$\epsilon_{rr} = \frac{1}{2} \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{2} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{2} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{2} \frac{\partial^2 \psi}{\partial \theta^2}, \quad 2\epsilon_{r\theta} = \frac{1}{2} \frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{2} \frac{\partial^2 \phi}{\partial \theta \partial r} - \frac{1}{2} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{2} \frac{\partial^2 \psi}{\partial \theta^2}$$

RELATIONS ARE FOUND BY SUBSTITUTING EQS. (4) INTO EQS. (3).

7) FIELD EQUATIONS:

$$\begin{aligned} \nabla^2 u_r - \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + [\lambda_e + \mu_e + (\lambda_v + \mu_v) \frac{\partial}{\partial t}] \frac{\partial \phi}{\partial r} - \rho \frac{\partial^2 u_r}{\partial t^2} \\ \nabla^2 u_\theta - \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} + [\lambda_e + \mu_e + (\lambda_v + \mu_v) \frac{\partial}{\partial t}] \frac{\partial \phi}{\partial \theta} - \rho \frac{\partial^2 u_\theta}{\partial t^2} \\ = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \end{aligned}$$

FORM (4) IS NOT AS USEFUL.

7) FIELD EQUATIONS:

$$\begin{aligned} (3\alpha + 2\beta + \frac{\partial}{\partial t}) [\nabla^2 u_r - \frac{u_r}{r} - \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}] + [\alpha + \frac{\lambda_v + \mu_v}{\rho} (2\beta + \frac{\partial}{\partial t})] \frac{\partial \phi}{\partial r} \\ = \frac{1}{\rho} [(3\alpha + 4\beta + \frac{\partial}{\partial t}) \frac{\partial^2 \phi}{\partial r^2} + 2\beta (2\alpha + 3\alpha) \frac{\partial^2 \phi}{\partial \theta^2}] \\ (3\alpha + 2\beta + \frac{\partial}{\partial t}) [\nabla^2 u_\theta - \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta}] + [\alpha + \frac{\lambda_v + \mu_v}{\rho} (2\beta + \frac{\partial}{\partial t})] \frac{\partial \phi}{\partial \theta} \\ = \frac{1}{\rho} [(3\alpha + 4\beta + \frac{\partial}{\partial t}) \frac{\partial^2 \phi}{\partial r \partial \theta} + 2\beta (2\alpha + 3\alpha) \frac{\partial^2 \phi}{\partial \theta^2}] \end{aligned}$$

WHERE Δ AND ∇^2 ARE GIVEN ABOVE IN (4) & (5). ($\alpha = \mu_e/\rho$)

FORM (4) IS NOT AS USEFUL.

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$$\begin{aligned} \epsilon_{rr} &= \Delta + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{rr} \\ \epsilon_{\theta\theta} &= \Delta + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{\theta\theta} \\ \epsilon_{zz} &= 0 \end{aligned}$$

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$$\begin{aligned} \epsilon_{rr} &= M_1 \{ \lambda_e \nabla^2 \phi + 2\mu_e [\frac{\partial^2 \phi}{\partial r^2} - \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial \phi}{\partial r}) + \frac{\partial^2 \phi}{\partial \theta^2}] \} + N_1 \\ \epsilon_{\theta\theta} &= M_1 \{ \lambda_e \nabla^2 \phi + 2\mu_e [\frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial \phi}{\partial r})] \} + N_1 \\ \epsilon_{zz} &= M_1 \{ \lambda_e \nabla^2 \phi \} + N_1, \quad \epsilon_{r\theta} = 0, \quad \epsilon_{rz} = 0, \\ \epsilon_{\theta z} &= 0 \end{aligned}$$

WHERE, $M_1 = \frac{\partial}{\partial t} (2\beta - \frac{\partial}{\partial t})$ (TO HAVE THESE IN TERMS OF ϕ & ψ , WE REPLACE $\nabla^2 \chi$ BY $-\psi$)

$$N_1 = \frac{[\alpha(3\lambda_e + 2\mu_e)]}{(3\alpha + 2\beta) \frac{\partial}{\partial t}} \left[\frac{(3\alpha + 4\beta) \frac{\partial^2 \phi}{\partial r^2} + 2\beta [(3\alpha + 2\beta) \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2}]}{4\beta^2 - \frac{\partial^2}{\partial t^2}} \right] \nabla^2 \phi$$

9) TELEGRAPH EQUATIONS FOR THE SCALAR & VECTOR POTENTIALS:

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

9) TELEGRAPH EQUATIONS FOR THE SCALAR & VECTOR POTENTIALS:

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

1) RELATION BETWEEN THE TRAC

$$\epsilon_{rr} = [(3\lambda_e + 2\mu_e) \epsilon_{rr} + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{rr}]$$

$$\epsilon_{rr} = \text{BULK OR HYDROSTATIC ST}$$

2) CONSTITUTIVE RELATIONS:

$$\begin{aligned} \epsilon_{rr} &= (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \epsilon_{rr} + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{rr} \\ \epsilon_{\theta\theta} &= (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \epsilon_{\theta\theta} + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{\theta\theta} \\ \epsilon_{zz} &= (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \epsilon_{zz} + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{zz} \\ \epsilon_{r\theta} &= 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{r\theta} \end{aligned}$$

3) STRAIN-DISPLACEMENT RELATIONS:

$$\epsilon_{rr} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \quad \epsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \quad \epsilon_{zz} = 0$$

4) DISPLACEMENT-INDEPENDENT POTENTIALS:

$$\begin{aligned} u_r &= \frac{\partial}{\partial r} [\phi + \frac{\partial}{\partial r} (r\psi)] - r \nabla^2 \psi \\ u_\theta &= \frac{1}{r} \frac{\partial}{\partial \theta} [\phi + \frac{\partial}{\partial r} (r\psi)] - \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2} \\ u_\phi &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} [\phi + \frac{\partial}{\partial r} (r\psi)] + \frac{\partial^2 \psi}{\partial \theta^2} \end{aligned}$$

5) STRESS-DISPLACEMENT RELATIONS:

$$\epsilon_{rr} = (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \epsilon_{rr} + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{rr}$$

WHERE THE ϵ_{ij} ARE GIVEN ABOVE

6) STRAIN-INDEPENDENT POTENTIALS:

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial^2}{\partial r^2} [\phi + \frac{\partial}{\partial r} (r\psi)] + \frac{\partial}{\partial r} [r \nabla^2 \psi] = \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \psi}{\partial r^2} \\ \epsilon_{\theta\theta} &= [\frac{1}{r} \frac{\partial}{\partial \theta} + \frac{1}{r} \frac{\partial^2}{\partial \theta^2}] [\phi + \frac{\partial}{\partial r} (r\psi)] - \nabla^2 \psi \end{aligned}$$

$$\begin{aligned} \epsilon_{zz} &= \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2} \\ 2\epsilon_{r\theta} &= \frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2} \\ 2\epsilon_{r\phi} &= \frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2} \\ 2\epsilon_{\theta z} &= \frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2} \end{aligned}$$

THESE ARE FOUND BY SUBSTITUTING EQS. (4) INTO EQS. (3).

7) FIELD EQUATIONS:

$$\begin{aligned} (\mu_e + \mu_v \frac{\partial}{\partial t}) \nabla^2 \phi + (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \nabla^2 \psi \\ \left\{ \begin{aligned} [\mu_e + \mu_v \frac{\partial}{\partial t}] [\nabla^2 u_r - \frac{u_r}{r} - \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}] - \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} \\ [\mu_e + \mu_v \frac{\partial}{\partial t}] [\nabla^2 u_\theta - \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta}] - \frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r \partial \theta} \\ [\mu_e + \mu_v \frac{\partial}{\partial t}] [\nabla^2 u_\phi - \frac{u_\phi}{r \sin \theta} - \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2}] - \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2} \end{aligned} \right\} \\ (\lambda_e + 2\mu_e + (\lambda_v + 2\mu_v) \frac{\partial}{\partial t}) \nabla^2 \psi \end{aligned}$$

THE DILATION Δ IS GIVEN IN (5)

8) STRESS-INDEPENDENT POTENTIALS:

$$\begin{aligned} \epsilon_{rr} &= (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \nabla^2 \phi + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{rr} \\ \epsilon_{\theta\theta} &= (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \nabla^2 \phi + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{\theta\theta} \\ \epsilon_{zz} &= (\lambda_e + \lambda_v \frac{\partial}{\partial t}) \nabla^2 \phi + 2(\mu_e + \mu_v \frac{\partial}{\partial t}) \epsilon_{zz} \\ \epsilon_{r\theta} &= (\mu_e + \mu_v \frac{\partial}{\partial t}) [\frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2}] \\ \epsilon_{r\phi} &= (\mu_e + \mu_v \frac{\partial}{\partial t}) [\frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2}] \\ \epsilon_{\theta z} &= (\mu_e + \mu_v \frac{\partial}{\partial t}) [\frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2}] \end{aligned}$$

9) TELEGRAPH EQUATIONS FOR

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

(THESE ARE EQS.
6) SUBSTITUTED
INTO EQ 6 2).

<p>NO GENERAL EXPRESSION CAN BE GIVEN VALID IN ANY COORD. SYSTEM.</p>	$2\epsilon_{\theta\theta} = \frac{2}{r} \frac{\partial^2}{\partial r \partial z} [\phi + \frac{\partial \psi}{\partial z}] - \frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \psi) + \frac{\partial^2}{\partial z^2} (\nabla^2 \chi) = \frac{2}{r} \frac{\partial^2 \psi}{\partial r \partial z} + \frac{1}{r} \frac{\partial^2 \chi}{\partial z^2}$ $2\epsilon_{rz} = 2 \frac{\partial^2}{\partial r \partial z} [\phi + \frac{\partial \psi}{\partial z}] - \frac{\partial}{\partial r} (\nabla^2 \psi) - \frac{1}{r} \frac{\partial^2}{\partial z \partial \theta} (\nabla^2 \chi) = 2 \frac{\partial^2 \psi}{\partial r \partial z} + \frac{\partial}{\partial r} (\frac{\partial^2 \psi}{\partial z^2})$ $2\epsilon_{\theta z} = 2 \frac{\partial^2}{\partial r \partial \theta} [\frac{1}{r} [\phi + \frac{\partial \psi}{\partial z}]] + [\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}] (\nabla^2 \chi) = 2 \frac{\partial}{\partial r} [\frac{1}{r} \frac{\partial \psi}{\partial \theta}] +$	<p>7) FIELD EQUATIONS:</p> <p>i) $[\mu_e + \mu_v \frac{\partial}{\partial t}] \nabla^2 \vec{u} + [\lambda_e + \mu_e + (\lambda_v + \mu_v) \frac{\partial}{\partial t}] \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$</p> <p>OR, IN COMPONENT FORM:</p> $[\mu_e + \mu_v \frac{\partial}{\partial t}] [\nabla^2 u_r - \frac{u_r}{r} - \frac{\partial}{\partial r} \frac{\partial u_\theta}{\partial \theta}] + [\lambda_e + \mu_e + (\lambda_v + \mu_v) \frac{\partial}{\partial t}] \frac{\partial \Delta}{\partial r} = \rho \frac{\partial^2 u_r}{\partial t^2}$ $[\mu_e + \mu_v \frac{\partial}{\partial t}] [\nabla^2 u_\theta - \frac{u_\theta}{r} + \frac{\partial}{\partial r} \frac{\partial u_r}{\partial \theta}] + [\lambda_e + \mu_e + (\lambda_v + \mu_v) \frac{\partial}{\partial t}] \frac{1}{r} \frac{\partial \Delta}{\partial \theta} = \rho \frac{\partial^2 u_\theta}{\partial t^2}$ $[\mu_e + \mu_v \frac{\partial}{\partial t}] [\nabla^2 u_z] + [\lambda_e + \mu_e + (\lambda_v + \mu_v) \frac{\partial}{\partial t}] \frac{\partial \Delta}{\partial z} = \rho \frac{\partial^2 u_z}{\partial t^2}$ <p>WHERE Δ AND ∇^2 ARE GIVEN ABOVE IN 4) AND 5).</p> <p>ii) $[\lambda_e + 2\mu_e + (\lambda_v + 2\mu_v) \frac{\partial}{\partial t}] \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - (\mu_e + \mu_v \frac{\partial}{\partial t}) \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$</p> <p>THESE TWO FORMS i) AND ii) ARE EQUIVALENT. ∇^2 GIVEN IN (4).</p>
<p>7) THE FIELD EQUATIONS: (FOR THE DISPLACEMENT FIELD)</p> <p>i):</p> $[QR + 2S(P+Q)] \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) + Q(3R+2S) \nabla^2 \vec{u} = 2S(3R+2S) \rho \ddot{\vec{u}}$ <p>ii):</p> $2[2QR+S(P+2Q)] \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - Q(3R+2S) \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) + 2S(3R+2S) \rho \ddot{\vec{u}}$ <p>NOTE: $\nabla^2 \vec{u} = \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{u})$</p> <p>Q, R, S, P ARE OPERATORS DEPENDING ON THE MODEL.</p>	<p>7) FIELD EQUATIONS:</p> <p>i): $[\mu_e + \mu_v \frac{\partial}{\partial t}] \nabla^2 \vec{u} + [\lambda_e + \mu_e + (\lambda_v + \mu_v) \frac{\partial}{\partial t}] \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$</p> <p>OR, IN COMPONENT FORM:</p> $[\mu_e + \mu_v \frac{\partial}{\partial t}] [\nabla^2 u_r - \frac{u_r}{r} - \frac{\partial}{\partial r} \frac{\partial u_\theta}{\partial \theta}] + [\lambda_e + \mu_e + (\lambda_v + \mu_v) \frac{\partial}{\partial t}] \frac{\partial \Delta}{\partial r} = \rho \frac{\partial^2 u_r}{\partial t^2}$ $[\mu_e + \mu_v \frac{\partial}{\partial t}] [\nabla^2 u_\theta - \frac{u_\theta}{r} + \frac{\partial}{\partial r} \frac{\partial u_r}{\partial \theta}] + [\lambda_e + \mu_e + (\lambda_v + \mu_v) \frac{\partial}{\partial t}] \frac{1}{r} \frac{\partial \Delta}{\partial \theta} = \rho \frac{\partial^2 u_\theta}{\partial t^2}$ $[\mu_e + \mu_v \frac{\partial}{\partial t}] [\nabla^2 u_z] + [\lambda_e + \mu_e + (\lambda_v + \mu_v) \frac{\partial}{\partial t}] \frac{\partial \Delta}{\partial z} = \rho \frac{\partial^2 u_z}{\partial t^2}$ <p>WHERE Δ AND ∇^2 ARE GIVEN ABOVE IN 4) AND 5).</p> <p>ii): $[\lambda_e + 2\mu_e + (\lambda_v + 2\mu_v) \frac{\partial}{\partial t}] \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - (\mu_e + \mu_v \frac{\partial}{\partial t}) \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$</p> <p>THESE TWO FORMS i) AND ii) ARE EQUIVALENT. ∇^2 GIVEN IN (4).</p>	<p>7) FIELD</p> <p>i) $[1 + \frac{\partial}{\partial t}] \nabla^2 \vec{u} = [\nabla^2 \vec{u}]$</p> <p>THIS CASE</p> <p>ii) $[1 + \frac{\partial}{\partial t}] \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) + \frac{\partial}{\partial t} \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \vec{\nabla} \times (\vec{\nabla} \times \vec{u})$</p> <p>THESE TWO</p>
<p>8) STRESS-INDEPENDENT POTENTIAL RELATIONS:</p> <p>NO GENERAL EXPRESSION CAN BE GIVEN VALID IN ANY COORD. SYSTEM.</p>	<p>8) STRESS-INDEPENDENT POTENTIAL RELATIONS:</p> $\tau_{rr} = [\lambda_e + \lambda_v \frac{\partial}{\partial t}] \nabla^2 \phi + 2[\mu_e + \mu_v \frac{\partial}{\partial t}] [\frac{\partial^2}{\partial r^2} (\phi + \frac{\partial \psi}{\partial z}) - \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi))]$ $\tau_{\theta\theta} = [\lambda_e + \lambda_v \frac{\partial}{\partial t}] \nabla^2 \phi + 2[\mu_e + \mu_v \frac{\partial}{\partial t}] [\frac{\partial^2}{\partial \theta^2} (\phi + \frac{\partial \psi}{\partial z}) + \frac{\partial}{\partial \theta} (\frac{1}{r} \frac{\partial}{\partial r} (\nabla^2 \chi))]$ $\tau_{zz} = [\lambda_e + \lambda_v \frac{\partial}{\partial t}] \nabla^2 \phi + 2[\mu_e + \mu_v \frac{\partial}{\partial t}] [\frac{\partial^2}{\partial z^2} (\phi + \frac{\partial \psi}{\partial z}) - \frac{\partial}{\partial z} (\nabla^2 \chi)]$ $\tau_{rz} = [\mu_e + \mu_v \frac{\partial}{\partial t}] [\frac{\partial}{\partial r} \frac{\partial^2}{\partial z^2} (\phi + \frac{\partial \psi}{\partial z}) - \frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \psi) + \frac{\partial^2}{\partial r \partial z} (\nabla^2 \chi)]$ $\tau_{r\theta} = [\mu_e + \mu_v \frac{\partial}{\partial t}] [\frac{\partial}{\partial r} \frac{\partial^2}{\partial \theta^2} (\phi + \frac{\partial \psi}{\partial z}) - \frac{\partial}{\partial r} (\nabla^2 \psi) - \frac{1}{r} \frac{\partial^2}{\partial \theta^2} (\nabla^2 \chi)]$ $\tau_{\theta z} = [\mu_e + \mu_v \frac{\partial}{\partial t}] [\frac{\partial}{\partial \theta} \frac{\partial^2}{\partial z^2} (\phi + \frac{\partial \psi}{\partial z}) + (\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}) \nabla^2 \chi]$ <p>THESE ARE EQS. (5) WITH $\Delta = \nabla^2 \phi$ AND ϵ_{ij} AS GIVEN IN EQS. (6)</p>	<p>8) STRESSES</p> <p>$\tau_{rr} = M_r \dot{\epsilon}_{rr}$</p> <p>$\tau_{\theta\theta} = M_\theta \dot{\epsilon}_{\theta\theta}$</p> <p>$\tau_{zz} = M_z \dot{\epsilon}_{zz}$</p> <p>$\tau_{r\theta} = M_{r\theta} \dot{\epsilon}_{r\theta}$</p> <p>$\tau_{\theta z} = M_{\theta z} \dot{\epsilon}_{\theta z}$</p> <p>$\tau_{rz} = M_{rz} \dot{\epsilon}_{rz}$</p> <p>$M_r = \frac{2\mu_e(2\mu_v + \lambda_v)}{4\mu_e + \lambda_v}$</p> <p>THESE ARE</p>
<p>9) TELEGRAPH-TYPE EQS. FOR SCALAR & VECTOR POTENTIALS.</p> <p>i) $[4QR + 2S(P+2Q)] \nabla^2 \phi = 2S(3R+2S) \rho \frac{\partial^2 \phi}{\partial t^2}$</p> <p>ii) $-Q[\vec{\nabla} \times (\vec{\nabla} \times \vec{\psi})] = 2S \rho \frac{\partial^2 \vec{\psi}}{\partial t^2}$</p> <p>EQ. ii) CAN ALSO BE WRITTEN AS FOLLOWS:</p> $\nabla^2 \vec{\psi} = \frac{2S \rho}{Q} \frac{\partial^2 \vec{\psi}}{\partial t^2} \quad \text{AND} \quad \vec{\nabla} \cdot \vec{\psi} = 0.$	<p>9) TELEGRAPH-TYPE EQS. FOR THE SCALAR & VECTOR POTENTIALS.</p> <p>i) $[1 + M \frac{\partial}{\partial t}] \nabla^2 \phi = \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2}$, $M = \frac{\lambda_v + 2\mu_v}{\lambda_e + 2\mu_e}$, $C^2 = \frac{\lambda_e + 2\mu_e}{\rho}$</p> <p>ii) $-[1 + N \frac{\partial}{\partial t}] \vec{\nabla} \times (\vec{\nabla} \times \vec{\psi}) = \frac{1}{C^2} \frac{\partial^2 \vec{\psi}}{\partial t^2}$, $N = \frac{\mu_v}{\mu_e}$, $C^2 = \frac{\mu_e}{\rho}$</p> <p>EQ. ii) CAN ALSO BE WRITTEN AS FOLLOWS:</p> $[1 + N \frac{\partial}{\partial t}] \nabla^2 \vec{\psi} = \frac{1}{C^2} \frac{\partial^2 \vec{\psi}}{\partial t^2} \quad \text{AND} \quad \vec{\nabla} \cdot \vec{\psi} = 0.$	<p>9) TELEGRAPH</p> <p>i) $[2\mu_e + \lambda_e \frac{\partial}{\partial t}] \nabla^2 \phi = \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2}$</p> <p>ii) $[2\mu_e + \lambda_e \frac{\partial}{\partial t}] \vec{\nabla} \times (\vec{\nabla} \times \vec{\psi}) = \frac{1}{C^2} \frac{\partial^2 \vec{\psi}}{\partial t^2}$</p> <p>EQ. ii)</p> <p>$\vec{\nabla} \cdot \vec{\psi} = 0$</p>
<p>10) TELEGRAPH-TYPE EQS. FOR INDEP. SCALAR POTENTIALS:</p> <p>i) $[4QR + 2S(P+2Q)] \nabla^2 \phi = 2S(3R+2S) \rho \frac{\partial^2 \phi}{\partial t^2}$</p> <p>ii) $Q \nabla^2 \psi = 2S \rho \frac{\partial^2 \psi}{\partial t^2}$</p> <p>iii) $Q \nabla^2 \chi = 2S \rho \frac{\partial^2 \chi}{\partial t^2}$</p>	<p>10) TELEGRAPH EQS. FOR INDEPENDENT SCALAR POTENTIALS:</p> <p>i) $[1 + M \frac{\partial}{\partial t}] \nabla^2 \phi = \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2}$</p> <p>ii) $[1 + N \frac{\partial}{\partial t}] \nabla^2 \psi = \frac{1}{C^2} \frac{\partial^2 \psi}{\partial t^2}$, $\lambda_i) [1 + N \frac{\partial}{\partial t}] \nabla^2 \chi = \frac{1}{C^2} \frac{\partial^2 \chi}{\partial t^2}$</p> <p>WHERE, $M = \frac{\lambda_v + 2\mu_v}{\lambda_e + 2\mu_e}$, $N = \frac{\mu_v}{\mu_e}$, $C^2 = \frac{\lambda_e + 2\mu_e}{\rho}$, $C^2 = \frac{\mu_e}{\rho}$</p>	<p>10) TELEGRAPH</p> <p>i) $[2\mu_e + \lambda_e \frac{\partial}{\partial t}] \nabla^2 \phi = \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2}$</p> <p>ii) $\nabla^2 \psi = \frac{1}{C^2} \frac{\partial^2 \psi}{\partial t^2}$</p> <p>FOR $C^2 = \frac{\mu_e}{\rho}$</p>
<p>11) SOLENOIDAL SOLUTION OF VECTOR TELEGRAPH-EQN.</p> <p>THE SOLUTION EXISTS ONLY IN FIVE COORDINATE SYSTEMS (CYLINDRICAL & SPHERICALS INCLUDED) BUT IT VARIES WITH THE SYSTEM USED, AND NO EXPRESSION EXISTS VALID FOR ALL SYSTEMS. NO GENERAL EXPRESSIONS FOR RELATIONS (4), (6) OR (8) ABOVE CAN BE GIVEN FOR THIS REASON.</p> <p>(SEE: P. MORSE & H. FESHBACH "METHODS OF THEORETICAL PHYSICS" VOL. 2, CHAPTER 13. MCGRAW-HILL, 1953)</p>	<p>11) SOLENOIDAL SOLUTION OF THE VECTOR TELEGRAPH-EQUATION.</p> <p>IT CAN BE SHOWN THAT THE SOLENOIDAL (i.e. $\vec{\nabla} \cdot \vec{\psi} = 0$) SOLUTION ($\vec{\psi} = \frac{1}{C^2} \frac{\partial^2 \vec{\psi}}{\partial t^2}$) IS THE SOLUTION OF $[1 + N \frac{\partial}{\partial t}] \vec{\nabla} \times (\vec{\nabla} \times \vec{\psi}) + \frac{1}{C^2} \frac{\partial^2 \vec{\psi}}{\partial t^2} = 0$ AND</p> $\vec{\psi} = \vec{\nabla} \times (\psi \vec{e}_r) + \vec{\nabla} \times (\chi \vec{e}_z) = (\vec{\nabla} \psi) \times \vec{e}_r - \vec{\nabla} (\frac{\partial \chi}{\partial z})$ <p>WHERE $\psi(r, \theta, z, t)$ AND $\chi(r, \theta, z, t)$ ARE TWO NEW SCALAR FUNCTIONS</p> <p>EQUATIONS i.e. $[1 + N \frac{\partial}{\partial t}] \nabla^2 \psi = \frac{1}{C^2} \frac{\partial^2 \psi}{\partial t^2}$ AND $[1 + N \frac{\partial}{\partial t}] \nabla^2 \chi = \frac{1}{C^2} \frac{\partial^2 \chi}{\partial t^2}$</p> $\psi_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} - \frac{\partial^2 \psi}{\partial r^2} \chi, \quad \psi_\theta = -\frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2} \chi, \quad \psi_z = -\frac{\partial \chi}{\partial z}$ <p>WHICH RELATE THE THREE CYLINDRICAL COMPONENTS $\psi_r, \psi_\theta, \psi_z$ OF $\vec{\psi}$ AND χ. SUBSTITUTING $\vec{\psi}$ FROM THE SOLUTION (A) INTO (8) THE CYLINDRICAL COMPONENTS OF THE DISPLACEMENT FIELD AND THE SC AS GIVEN ABOVE IN EQS. (4). THE SOLUTION (A) OR (B) INVOLVES ON WAVE-EQ. (OR TELEGRAPH-EQ. WITHOUT DAMPING) PROVIDED THAT ψ A</p>	<p>11) TELEGRAPH</p> <p>i) $[2\mu_e + \lambda_e \frac{\partial}{\partial t}] \nabla^2 \psi = \frac{1}{C^2} \frac{\partial^2 \psi}{\partial t^2}$</p> <p>ii) $[2\mu_e + \lambda_e \frac{\partial}{\partial t}] \nabla^2 \chi = \frac{1}{C^2} \frac{\partial^2 \chi}{\partial t^2}$</p> <p>FOR $C^2 = \frac{\mu_e}{\rho}$</p>
<p>12) REMARKS:</p> <p>SEE THE COMPARTMENTS TO THE RIGHT.</p>	<p>REMARKS:</p> <p>a) THE BASIC RESULT OF THIS TABLE, AND THAT IS WHY IT WAS ORGANIZED SATISFIED IN EACH ONE OF THE CASES CONSIDERED AND THEN THE DIS BOUNDARY VALUE PROBLEMS, AND WHICH IS COMMON IN OTHER DISC THE HEADER FAMILIAR WITH ELECTRODYNAMIC THEORY WILL QUICKLY N</p> <p>b) IT IS EVIDENT THAT IN ALL THE CASES IN THIS CHART WE RECOVER</p> <p>c) IN CYLINDRICAL COORDS. THE PLANE-STRESS RESULTS CAN BE QUICKLY</p> <p>i) THE CASE OF PLANE-STRAIN (OR PLANE-STRESS IN VIEW OF c) ABOVE) WILL</p> <p>c) A CASE OF SOME INTEREST, WHICH REDUCES THE TWO VISCOUS CONSTAN</p>	<p>12) REMARKS:</p> <p>SEE THE COMPARTMENTS TO THE RIGHT.</p>

<p>2</p> <p>1</p> <p>0</p> <p>1</p> <p>2</p> <p>3</p> <p>4</p> <p>5</p> <p>6</p> <p>7</p> <p>8</p> <p>9</p> <p>10</p> <p>11</p> <p>12</p> <p>13</p> <p>14</p> <p>15</p> <p>16</p> <p>17</p> <p>18</p> <p>19</p> <p>20</p> <p>21</p> <p>22</p> <p>23</p> <p>24</p> <p>25</p> <p>26</p> <p>27</p> <p>28</p> <p>29</p> <p>30</p> <p>31</p> <p>32</p> <p>33</p> <p>34</p> <p>35</p> <p>36</p> <p>37</p> <p>38</p> <p>39</p> <p>40</p> <p>41</p> <p>42</p> <p>43</p> <p>44</p> <p>45</p> <p>46</p> <p>47</p> <p>48</p> <p>49</p> <p>50</p> <p>51</p> <p>52</p> <p>53</p> <p>54</p> <p>55</p> <p>56</p> <p>57</p> <p>58</p> <p>59</p> <p>60</p> <p>61</p> <p>62</p> <p>63</p> <p>64</p> <p>65</p> <p>66</p> <p>67</p> <p>68</p> <p>69</p> <p>70</p> <p>71</p> <p>72</p> <p>73</p> <p>74</p> <p>75</p> <p>76</p> <p>77</p> <p>78</p> <p>79</p> <p>80</p> <p>81</p> <p>82</p> <p>83</p> <p>84</p> <p>85</p> <p>86</p> <p>87</p> <p>88</p> <p>89</p> <p>90</p> <p>91</p> <p>92</p> <p>93</p> <p>94</p> <p>95</p> <p>96</p> <p>97</p> <p>98</p> <p>99</p> <p>100</p>	<p>INTO EQS. (3).</p> $\frac{\partial^2 \psi}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\frac{\partial^2 \psi}{\partial r^2} - \nabla^2 \psi \right) \right] + \frac{\partial}{\partial r} \left(\frac{\partial^2 \chi}{\partial r^2} \right)$ $\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial}{\partial r} \left[2 \frac{\partial^2 \psi}{\partial r^2} - \nabla^2 \psi \right] - \frac{1}{r} \frac{\partial^2}{\partial \theta^2} (\nabla^2 \chi)$ $= 2 \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial \psi}{\partial r} \right] + \frac{2}{r} \frac{\partial}{\partial \theta} \left[\frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right] + \left[\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] (\nabla^2 \chi)$	$C_{rr} = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2}$ $C_{\theta\theta} = \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2}$ <p>THESE RELATIONS ARE FOUND BY SUBSTITUTING EQS. (4) INTO EQS. (3).</p>	<p>7) FIELD EQUATIONS:</p> <p>1) $[1 + \frac{1}{2\beta} \frac{\partial}{\partial t}] [\nabla^2 \tilde{u} + \frac{1}{2\beta} \frac{\partial}{\partial t} \nabla(\tilde{v} \cdot \tilde{u}) - \frac{1}{2\beta} \frac{\partial^2 \tilde{u}}{\partial t^2}] + \frac{3\alpha}{2\beta} [\nabla^2 \tilde{u} + \frac{1}{3} \nabla(\tilde{v} \cdot \tilde{u}) - \frac{1}{3} \frac{\partial^2 \tilde{u}}{\partial t^2}] = \frac{1}{2} \frac{\partial^2 \tilde{u}}{\partial t^2} + \frac{3\alpha + 2\beta}{2\beta} \frac{\partial \tilde{u}}{\partial t}$ <p>THIS CAN BE OPENED UP INTO COMPONENTS BY MEANS OF</p> $\nabla^2 \tilde{u} = [\nabla^2 u_r - \frac{u_r}{r} - \frac{2}{r} \frac{\partial u_\theta}{\partial \theta}] \hat{e}_r + [\nabla^2 u_\theta - \frac{u_\theta}{r} + \frac{2}{r} \frac{\partial u_r}{\partial \theta}] \hat{e}_\theta + [\nabla^2 u_z] \hat{e}_z$ $\nabla(\tilde{v} \cdot \tilde{u}) = \frac{\partial \Delta}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \Delta}{\partial \theta} \hat{e}_\theta + \frac{\partial \Delta}{\partial z} \hat{e}_z \quad \Delta \text{ AS GIVEN ABOVE IN (5).}$ <p>ii) $[1 + \frac{1}{2\beta} \frac{\partial}{\partial t}] [\frac{1}{2} \nabla(\tilde{v} \cdot \tilde{u}) - \frac{1}{2} \frac{\partial^2 \tilde{u}}{\partial t^2}] + \frac{3\alpha}{2\beta} [\frac{1}{3} \nabla(\tilde{v} \cdot \tilde{u}) - \frac{1}{3} \frac{\partial^2 \tilde{u}}{\partial t^2}] = \frac{1}{2} \frac{\partial^2 \tilde{u}}{\partial t^2} + (3\alpha + 2\beta) \frac{\partial \tilde{u}}{\partial t}$ <p>THESE TWO FORMS (i) AND (ii) ARE EQUIVALENT. ∇^2 GIVEN IN (4).</p> </p></p>	<p>7) FIELD EQUATIONS:</p> <p>1) $[1 + \frac{1}{2\beta} \frac{\partial}{\partial t}] [\nabla^2 u_r - \frac{u_r}{r} - \frac{2}{r} \frac{\partial u_\theta}{\partial \theta}] + [\lambda_0 + \mu_0 + (\lambda_0 + \mu_0) \frac{\partial}{\partial t}] \frac{\partial \Delta}{\partial r} - \rho \frac{\partial^2 u_r}{\partial t^2}$ $[1 + \frac{1}{2\beta} \frac{\partial}{\partial t}] [\nabla^2 u_\theta - \frac{u_\theta}{r} + \frac{2}{r} \frac{\partial u_r}{\partial \theta}] + [\lambda_0 + \mu_0 + (\lambda_0 + \mu_0) \frac{\partial}{\partial t}] \frac{1}{r} \frac{\partial \Delta}{\partial \theta} - \rho \frac{\partial^2 u_\theta}{\partial t^2}$ <p>WHERE, $\Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}$, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$.</p> <p>THE SECOND FORM (ii) IS NOT AS USEFUL.</p> <p>WHERE Δ A FORM (4)</p> </p>
<p>1</p> <p>2</p> <p>3</p> <p>4</p> <p>5</p> <p>6</p> <p>7</p> <p>8</p> <p>9</p> <p>10</p> <p>11</p> <p>12</p> <p>13</p> <p>14</p> <p>15</p> <p>16</p> <p>17</p> <p>18</p> <p>19</p> <p>20</p> <p>21</p> <p>22</p> <p>23</p> <p>24</p> <p>25</p> <p>26</p> <p>27</p> <p>28</p> <p>29</p> <p>30</p> <p>31</p> <p>32</p> <p>33</p> <p>34</p> <p>35</p> <p>36</p> <p>37</p> <p>38</p> <p>39</p> <p>40</p> <p>41</p> <p>42</p> <p>43</p> <p>44</p> <p>45</p> <p>46</p> <p>47</p> <p>48</p> <p>49</p> <p>50</p> <p>51</p> <p>52</p> <p>53</p> <p>54</p> <p>55</p> <p>56</p> <p>57</p> <p>58</p> <p>59</p> <p>60</p> <p>61</p> <p>62</p> <p>63</p> <p>64</p> <p>65</p> <p>66</p> <p>67</p> <p>68</p> <p>69</p> <p>70</p> <p>71</p> <p>72</p> <p>73</p> <p>74</p> <p>75</p> <p>76</p> <p>77</p> <p>78</p> <p>79</p> <p>80</p> <p>81</p> <p>82</p> <p>83</p> <p>84</p> <p>85</p> <p>86</p> <p>87</p> <p>88</p> <p>89</p> <p>90</p> <p>91</p> <p>92</p> <p>93</p> <p>94</p> <p>95</p> <p>96</p> <p>97</p> <p>98</p> <p>99</p> <p>100</p>	<p>8) STRESS-INDEPENDENT POTENTIAL RELATIONS:</p> $C_{rr} = M_1 \{ \lambda_0 \nabla^2 \varphi + 2\mu_0 [\frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi))] \} + N_1$ $C_{\theta\theta} = M_1 \{ \lambda_0 \nabla^2 \varphi + 2\mu_0 [\frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) + \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi))] \} + N_1$ $C_{zz} = M_1 \{ \lambda_0 \nabla^2 \varphi + 2\mu_0 [\frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{\partial}{\partial z} (\nabla^2 \psi)] \} + N_1$ $C_{r\theta} = M_1 \mu_0 [\frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \psi) + \frac{\partial^2}{\partial r \partial \theta} (\nabla^2 \chi)]$ $C_{rz} = M_1 \mu_0 [2 \frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{\partial}{\partial r} (\nabla^2 \psi) - \frac{1}{r} \frac{\partial^2}{\partial \theta^2} (\nabla^2 \chi)]$ $C_{\theta z} = M_1 \mu_0 [2 \frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) + \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial \psi}{\partial \theta}) - \frac{1}{r} \frac{\partial}{\partial \theta} (\frac{\partial^2}{\partial r^2} (\nabla^2 \chi))] \quad \text{WHERE,}$ $M_1 = \frac{2\beta(2\beta - 3\alpha)}{4\beta^2 - 3\alpha^2}, \quad N_1 = \frac{-\alpha(3\alpha + 2\mu_0) \{ - (3\alpha + 4\beta) \frac{\partial^2}{\partial r^2} + 2\beta [\frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi)) + \frac{1}{2\beta} \frac{\partial^2}{\partial t^2}] \}}{(3\alpha + 2\mu_0)^2 - \frac{\partial^2}{\partial t^2}}$ <p>THESE ARE EQS. (5) WITH $\Delta = \nabla^2 \varphi$ AND E_θ AS GIVEN IN EQS. (6).</p>	<p>8) STRESS-INDEPENDENT POTENTIAL RELATIONS:</p> $C_{rr} = [\lambda_0 + \lambda_0 \frac{\partial}{\partial t}] \nabla^2 \varphi + 2[\mu_0 + \mu_0 \frac{\partial}{\partial t}] [\frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi))]$ $C_{\theta\theta} = [\lambda_0 + \lambda_0 \frac{\partial}{\partial t}] \nabla^2 \varphi + 2[\mu_0 + \mu_0 \frac{\partial}{\partial t}] [\frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) + \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi))]$ $C_{zz} = [\lambda_0 + \lambda_0 \frac{\partial}{\partial t}] \nabla^2 \varphi$ $C_{r\theta} = [\mu_0 + \mu_0 \frac{\partial}{\partial t}] [2 \frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) + (\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}) \nabla^2 \chi]$ <p>TO HAVE THESE RELATIONS IN TERMS OF φ AND ψ WE REPLACE $\nabla^2 \chi$ BY $-\psi$. ∇^2 IS GIVEN ABOVE IN (7). THESE ARE EQS. (5) WITH $\Delta = \nabla^2 \varphi$ AND E_θ AS GIVEN IN EQS. (6).</p>	<p>8) STRESS-INDEPENDENT POTENTIAL RELATIONS:</p> $C_{rr} = M_1 \{ \lambda_0 \nabla^2 \varphi + 2\mu_0 [\frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi))] \} + N_1$ $C_{\theta\theta} = M_1 \{ \lambda_0 \nabla^2 \varphi + 2\mu_0 [\frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) + \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi))] \} + N_1$ $C_{zz} = M_1 \{ \lambda_0 \nabla^2 \varphi + 2\mu_0 [\frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{\partial}{\partial z} (\nabla^2 \psi)] \} + N_1$ $C_{r\theta} = M_1 \mu_0 [\frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \psi) + \frac{\partial^2}{\partial r \partial \theta} (\nabla^2 \chi)]$ $C_{rz} = M_1 \mu_0 [2 \frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) - \frac{\partial}{\partial r} (\nabla^2 \psi) - \frac{1}{r} \frac{\partial^2}{\partial \theta^2} (\nabla^2 \chi)]$ $C_{\theta z} = M_1 \mu_0 [2 \frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial \psi}{\partial t}) + \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial \psi}{\partial \theta}) - \frac{1}{r} \frac{\partial}{\partial \theta} (\frac{\partial^2}{\partial r^2} (\nabla^2 \chi))] \quad \text{WHERE,}$ $M_1 = \frac{2\beta(2\beta - 3\alpha)}{4\beta^2 - 3\alpha^2}, \quad N_1 = \frac{-\alpha(3\alpha + 2\mu_0) \{ - (3\alpha + 4\beta) \frac{\partial^2}{\partial r^2} + 2\beta [\frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi)) + \frac{1}{2\beta} \frac{\partial^2}{\partial t^2}] \}}{(3\alpha + 2\mu_0)^2 - \frac{\partial^2}{\partial t^2}}$	
<p>1</p> <p>2</p> <p>3</p> <p>4</p> <p>5</p> <p>6</p> <p>7</p> <p>8</p> <p>9</p> <p>10</p> <p>11</p> <p>12</p> <p>13</p> <p>14</p> <p>15</p> <p>16</p> <p>17</p> <p>18</p> <p>19</p> <p>20</p> <p>21</p> <p>22</p> <p>23</p> <p>24</p> <p>25</p> <p>26</p> <p>27</p> <p>28</p> <p>29</p> <p>30</p> <p>31</p> <p>32</p> <p>33</p> <p>34</p> <p>35</p> <p>36</p> <p>37</p> <p>38</p> <p>39</p> <p>40</p> <p>41</p> <p>42</p> <p>43</p> <p>44</p> <p>45</p> <p>46</p> <p>47</p> <p>48</p> <p>49</p> <p>50</p> <p>51</p> <p>52</p> <p>53</p> <p>54</p> <p>55</p> <p>56</p> <p>57</p> <p>58</p> <p>59</p> <p>60</p> <p>61</p> <p>62</p> <p>63</p> <p>64</p> <p>65</p> <p>66</p> <p>67</p> <p>68</p> <p>69</p> <p>70</p> <p>71</p> <p>72</p> <p>73</p> <p>74</p> <p>75</p> <p>76</p> <p>77</p> <p>78</p> <p>79</p> <p>80</p> <p>81</p> <p>82</p> <p>83</p> <p>84</p> <p>85</p> <p>86</p> <p>87</p> <p>88</p> <p>89</p> <p>90</p> <p>91</p> <p>92</p> <p>93</p> <p>94</p> <p>95</p> <p>96</p> <p>97</p> <p>98</p> <p>99</p> <p>100</p>	<p>9) TELEGRAPH-TYPE EQS. FOR THE SCALAR & VECTOR POTENTIALS:</p> <p>i) $[2\beta + 4\alpha \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}] \nabla^2 \varphi = \frac{1}{2} [(3\alpha + 4\beta) \frac{\partial^2}{\partial r^2} + 2\beta (\frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi)) + \frac{\partial^2}{\partial t^2})]$ <p>ii) $-\nabla \times (\nabla \times \tilde{\psi}) = \frac{1}{2} (2\beta + \frac{\partial}{\partial t}) \frac{\partial \tilde{\psi}}{\partial t}$ <p>EQ. (ii) CAN ALSO BE WRITTEN AS FOLLOWS,</p> $\nabla^2 \tilde{\psi} = \frac{1}{2} (2\beta + \frac{\partial}{\partial t}) \frac{\partial \tilde{\psi}}{\partial t} \quad \text{AND} \quad \nabla \cdot \tilde{\psi} = 0$ </p></p>	<p>9) TELEGRAPH EQS. FOR THE SCALAR & VECTOR POTENTIALS:</p> <p>i) $(1 + M \frac{\partial}{\partial t}) \nabla^2 \varphi = \frac{1}{2} \frac{\partial^2 \varphi}{\partial t^2}$, $ii) (1 + N \frac{\partial}{\partial t}) \nabla^2 \tilde{\psi} = \frac{1}{2} \frac{\partial^2 \tilde{\psi}}{\partial t^2}$</p> <p>WHERE, $M = \frac{1}{2} \frac{2\beta + 4\alpha}{\lambda_0 + 2\mu_0}$, $N = \frac{\mu_0}{\lambda_0}$, $C_1^2 = \frac{2\beta + 2\mu_0}{\lambda_0}$, $C_2^2 = \frac{\mu_0}{\lambda_0}$, $\nabla \cdot \tilde{\psi} = 0$</p> <p>HERE, $\tilde{\psi} = \tilde{e}_\theta \psi_\theta(r, \theta, t) = -\tilde{e}_z \nabla^2 \chi(r, \theta, t)$. HENCE, $\nabla \cdot \tilde{\psi} = 0$, AND EQ. (ii) CAN BE WRITTEN IN EITHER ALTERNATIVE FORM GIVEN IN (10).</p>	<p>9) TELEGRAPH-TYPE EQS. FOR THE SCALAR & VECTOR POTENTIALS:</p> <p>i) $[2\beta + 4\alpha \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}] \nabla^2 \varphi = \frac{1}{2} [(3\alpha + 4\beta) \frac{\partial^2}{\partial r^2} + 2\beta (\frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi)) + \frac{\partial^2}{\partial t^2})]$ <p>ii) $-\nabla \times (\nabla \times \tilde{\psi}) = \frac{1}{2} (2\beta + \frac{\partial}{\partial t}) \frac{\partial \tilde{\psi}}{\partial t}$ <p>EQ. (ii) CAN ALSO BE WRITTEN AS FOLLOWS,</p> $\nabla^2 \tilde{\psi} = \frac{1}{2} (2\beta + \frac{\partial}{\partial t}) \frac{\partial \tilde{\psi}}{\partial t} \quad \text{AND} \quad \nabla \cdot \tilde{\psi} = 0$ </p></p>	
<p>1</p> <p>2</p> <p>3</p> <p>4</p> <p>5</p> <p>6</p> <p>7</p> <p>8</p> <p>9</p> <p>10</p> <p>11</p> <p>12</p> <p>13</p> <p>14</p> <p>15</p> <p>16</p> <p>17</p> <p>18</p> <p>19</p> <p>20</p> <p>21</p> <p>22</p> <p>23</p> <p>24</p> <p>25</p> <p>26</p> <p>27</p> <p>28</p> <p>29</p> <p>30</p> <p>31</p> <p>32</p> <p>33</p> <p>34</p> <p>35</p> <p>36</p> <p>37</p> <p>38</p> <p>39</p> <p>40</p> <p>41</p> <p>42</p> <p>43</p> <p>44</p> <p>45</p> <p>46</p> <p>47</p> <p>48</p> <p>49</p> <p>50</p> <p>51</p> <p>52</p> <p>53</p> <p>54</p> <p>55</p> <p>56</p> <p>57</p> <p>58</p> <p>59</p> <p>60</p> <p>61</p> <p>62</p> <p>63</p> <p>64</p> <p>65</p> <p>66</p> <p>67</p> <p>68</p> <p>69</p> <p>70</p> <p>71</p> <p>72</p> <p>73</p> <p>74</p> <p>75</p> <p>76</p> <p>77</p> <p>78</p> <p>79</p> <p>80</p> <p>81</p> <p>82</p> <p>83</p> <p>84</p> <p>85</p> <p>86</p> <p>87</p> <p>88</p> <p>89</p> <p>90</p> <p>91</p> <p>92</p> <p>93</p> <p>94</p> <p>95</p> <p>96</p> <p>97</p> <p>98</p> <p>99</p> <p>100</p>	<p>10) TELEGRAPH EQS. FOR THE INDEPENDENT SCALAR POTENTIALS:</p> <p>i) $[2\beta + 4\alpha \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}] \nabla^2 \varphi = \frac{1}{2} [(3\alpha + 4\beta) \frac{\partial^2}{\partial r^2} + 2\beta (\frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi)) + \frac{\partial^2}{\partial t^2})]$ <p>ii) $\nabla^2 \psi = \frac{1}{2} (2\beta + \frac{\partial}{\partial t}) \frac{\partial \psi}{\partial t}$, $iii) \nabla^2 \chi = \frac{1}{2} (2\beta + \frac{\partial}{\partial t}) \frac{\partial \chi}{\partial t}$</p> <p>FOR $C_1^2 \ll C_2^2$ (i.e. RUBBER) EQ. (i) REDUCES TO:</p> $\nabla^2 \varphi = \frac{1}{2} [\frac{\partial^2}{\partial r^2} + (3\alpha + 2\beta) \frac{\partial}{\partial t}] \quad C_1^2 = \frac{2\beta + 2\mu_0}{\lambda_0}, \quad C_2^2 = \frac{\mu_0}{\lambda_0}$ </p>	<p>10) TELEGRAPH EQS. FOR THE INDEPENDENT SCALAR POTENTIALS:</p> <p>i) $(1 + M \frac{\partial}{\partial t}) \nabla^2 \varphi = \frac{1}{2} \frac{\partial^2 \varphi}{\partial t^2}$</p> <p>ii) $(1 + N \frac{\partial}{\partial t}) \nabla^2 \psi = \frac{1}{2} \frac{\partial^2 \psi}{\partial t^2}$ OR, $-\nabla^2 \{ (1 + N \frac{\partial}{\partial t}) \nabla^2 \chi - \frac{1}{2} \frac{\partial^2 \chi}{\partial t^2} \} = 0$</p> <p>WHERE M, N, C_1^2, C_2^2 ARE GIVEN ABOVE IN 9).</p>	<p>10) TELEGRAPH EQS. FOR THE INDEPENDENT SCALAR POTENTIALS:</p> <p>i) $[2\beta + 4\alpha \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}] \nabla^2 \varphi = \frac{1}{2} [(3\alpha + 4\beta) \frac{\partial^2}{\partial r^2} + 2\beta (\frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial}{\partial \theta} (\nabla^2 \chi)) + \frac{\partial^2}{\partial t^2})]$ <p>ii) $\nabla^2 \psi = \frac{1}{2} (2\beta + \frac{\partial}{\partial t}) \frac{\partial \psi}{\partial t}$, $iii) \nabla^2 \chi = \frac{1}{2} (2\beta + \frac{\partial}{\partial t}) \frac{\partial \chi}{\partial t}$</p> <p>FOR $C_1^2 \ll C_2^2$ (i.e. RUBBER) EQ. (i) REDUCES TO:</p> $\nabla^2 \varphi = \frac{1}{2} [\frac{\partial^2}{\partial r^2} + (3\alpha + 2\beta) \frac{\partial}{\partial t}] \quad C_1^2 = \frac{2\beta + 2\mu_0}{\lambda_0}, \quad C_2^2 = \frac{\mu_0}{\lambda_0}$ </p>	
<p>1</p> <p>2</p> <p>3</p> <p>4</p> <p>5</p> <p>6</p> <p>7</p> <p>8</p> <p>9</p> <p>10</p> <p>11</p> <p>12</p> <p>13</p> <p>14</p> <p>15</p> <p>16</p> <p>17</p> <p>18</p> <p>19</p> <p>20</p> <p>21</p> <p>22</p> <p>23</p> <p>24</p> <p>25</p> <p>26</p> <p>27</p> <p>28</p> <p>29</p> <p>30</p> <p>31</p> <p>32</p> <p>33</p> <p>34</p> <p>35</p> <p>36</p> <p>37</p> <p>38</p> <p>39</p> <p>40</p> <p>41</p> <p>42</p> <p>43</p> <p>44</p> <p>45</p> <p>46</p> <p>47</p> <p>48</p> <p>49</p> <p>50</p> <p>51</p> <p>52</p> <p>53</p> <p>54</p> <p>55</p> <p>56</p> <p>57</p> <p>58</p> <p>59</p> <p>60</p> <p>61</p> <p>62</p> <p>63</p> <p>64</p> <p>65</p> <p>66</p> <p>67</p> <p>68</p> <p>69</p> <p>70</p> <p>71</p> <p>72</p> <p>73</p> <p>74</p> <p>75</p> <p>76</p> <p>77</p> <p>78</p> <p>79</p> <p>80</p> <p>81</p> <p>82</p> <p>83</p> <p>84</p> <p>85</p> <p>86</p> <p>87</p> <p>88</p> <p>89</p> <p>90</p> <p>91</p> <p>92</p> <p>93</p> <p>94</p> <p>95</p> <p>96</p> <p>97</p> <p>98</p> <p>99</p> <p>100</p>	<p>11) EQUATION IN CYLINDRICAL COORDINATES:</p> <p>1) SOLUTION OF THE VECTOR TELEGRAPH EQUATION (VIZ, $(1 + N \frac{\partial}{\partial t}) \nabla^2 \tilde{\psi} = \frac{1}{2} \frac{\partial^2 \tilde{\psi}}{\partial t^2}$) AND THAT IT IS GIVEN BY EITHER ONE OF THE FOLLOWING,</p> <p>i) $\tilde{\psi} = \tilde{e}_\theta (\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r}) \chi$ (A) [CYLINDRICAL COORDS. ONLY],</p> <p>ii) $\tilde{\psi} = \tilde{e}_z \frac{\partial^2}{\partial r^2} \chi$ (B)</p> <p>iii) $\tilde{\psi} = \tilde{e}_\theta \frac{\partial^2}{\partial r^2} \chi - \frac{1}{r} \frac{\partial \chi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2}$ (C)</p> <p>iv) $\tilde{\psi} = \tilde{e}_\theta \frac{\partial^2}{\partial r^2} \chi - \frac{1}{r} \frac{\partial \chi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2}$ (D)</p> <p>THE SOLUTION (A) IN COMPONENT FORM IS,</p> $\psi_\theta = -\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2}$ <p>THE VECTOR POTENTIAL $\tilde{\psi}$ TO THE TWO SCALAR FUNCTIONS φ AND ψ INTO $\tilde{\psi} = \tilde{e}_\theta \varphi + \tilde{e}_z \psi$ YIELDS THE RELATIONS BETWEEN THE FIELD AND THE SCALAR FUNCTIONS (i.e. INDEPENDENT POTENTIALS) φ, ψ, χ INVOLVES ONLY SPACE OPERATORS, SO IT ALSO HOLDS FOR THE VECTOR POTENTIAL $\tilde{\psi}$ DIVIDED THAT ψ AND χ THEN SATISFY SCALAR WAVE-EQS. (i.e. $\nabla^2 \varphi = \frac{1}{2} \frac{\partial^2 \varphi}{\partial t^2}$, $\nabla^2 \psi = \frac{1}{2} \frac{\partial^2 \psi}{\partial t^2}$, $\nabla^2 \chi = \frac{1}{2} \frac{\partial^2 \chi}{\partial t^2}$)</p>	<p>11) SOLENOIDAL SOLUTION OF THE VECTOR TELEGRAPH EQUATION IN PLANE:</p> <p>THE SOLUTION FOLLOWS THAT GIVEN TO THE LEFT FOR GENERAL CYLINDRICAL COORDINATES. THEREFORE, ONE CAN ANALOGOUSLY SHOW THAT THE SOLENOIDAL SOLUTION OF THE VECTOR TELEGRAPH EQUATION IN PLANE IS,</p> $\tilde{\psi} = \tilde{e}_z \nabla^2 \chi$ <p>PROVIDED THAT $\chi(r, \theta, t)$ SATISFIES THE SCALAR TELEGRAPH EQUATION SINCE THE LAPLACIAN AND THE CURL OPERATORS ACT ON A θ-INDEPENDENT CASE.</p> <p>SOLUTION (A) IN COMPONENT-FORM IS: $\psi_\theta = 0$, $\psi_z = 0$, $\psi_r = -\nabla^2 \chi$ A θ-COMPONENT WHICH IS θ-INDEPENDENT. SUBSTITUTING SOLN. (A) IN POTENTIAL RELATIONS GIVEN IN EQS. (4) ABOVE. WE REPEAT THAT THE TELEGRAPH-EQUATION REDUCES TO THE WAVE-EQUATION, AS IT OCCURS IN SCALAR WAVE-EQUATION $[\nabla^2 - \frac{1}{2} \frac{\partial^2}{\partial t^2}] \chi = 0$.</p>	<p>11) EQUATION IN CYLINDRICAL COORDINATES:</p> <p>1) SOLUTION OF THE VECTOR TELEGRAPH EQUATION (VIZ, $(1 + N \frac{\partial}{\partial t}) \nabla^2 \tilde{\psi} = \frac{1}{2} \frac{\partial^2 \tilde{\psi}}{\partial t^2}$) AND THAT IT IS GIVEN BY EITHER ONE OF THE FOLLOWING,</p> <p>i) $\tilde{\psi} = \tilde{e}_\theta (\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r}) \chi$ (A) [CYLINDRICAL COORDS. ONLY],</p> <p>ii) $\tilde{\psi} = \tilde{e}_z \frac{\partial^2}{\partial r^2} \chi$ (B)</p> <p>iii) $\tilde{\psi} = \tilde{e}_\theta \frac{\partial^2}{\partial r^2} \chi - \frac{1}{r} \frac{\partial \chi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2}$ (C)</p> <p>iv) $\tilde{\psi} = \tilde{e}_\theta \frac{\partial^2}{\partial r^2} \chi - \frac{1}{r} \frac{\partial \chi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2}$ (D)</p> <p>THE SOLUTION (A) IN COMPONENT FORM IS,</p> $\psi_\theta = -\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2}$ <p>THE VECTOR POTENTIAL $\tilde{\psi}$ TO THE TWO SCALAR FUNCTIONS φ AND $\$</p>	

2. $\epsilon_{rr} = \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2}$, $2\epsilon_{\theta\theta} = \frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2}$

3). EQS. 4) INTO EQS. 3).

7) FIELD EQUATIONS:

i) $[3\alpha + 2\beta + \frac{3}{2}\epsilon] [\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r} \frac{\partial u_\theta}{\partial \theta}] + [\alpha + \frac{2\lambda + \mu}{\mu} (2\beta + \frac{3}{2}\epsilon)] \frac{\partial \theta}{\partial r} =$
 $= \frac{1}{\epsilon} [(3\alpha + 4\beta + \frac{3}{2}\epsilon) \frac{\partial^2 u_r}{\partial r^2} + 2\beta (2\beta + 3\alpha) \frac{\partial u_r}{\partial r}]$
 $[3\alpha + 2\beta + \frac{3}{2}\epsilon] [\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r} \frac{\partial u_r}{\partial \theta}] + [\alpha + \frac{2\lambda + \mu}{\mu} (2\beta + \frac{3}{2}\epsilon)] \frac{1}{r} \frac{\partial \theta}{\partial \theta} =$
 $= \frac{1}{\epsilon} [(3\alpha + 4\beta + \frac{3}{2}\epsilon) \frac{\partial^2 u_\theta}{\partial r^2} + 2\beta (2\beta + 3\alpha) \frac{\partial u_\theta}{\partial r}]$

WHERE Δ AND ∇^2 ARE GIVEN ABOVE IN (4) & (5). ($c_s^2 = \mu/\rho$)
 FORM (2) IS NOT AS USEFUL.

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$\begin{cases} \tau_{rr} = M_1 \{ \lambda_0 \nabla^2 \phi + 2\mu_0 [\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial \chi}{\partial \theta})] \} + N_1 \\ \tau_{\theta\theta} = M_1 \{ \lambda_0 \nabla^2 \phi + 2\mu_0 [\frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial \chi}{\partial \theta})] \} + N_1 \\ \tau_{r\theta} = M_1 \{ \lambda_0 \nabla^2 \phi + N_1 \} \quad \tau_{\theta r} = 0, \quad \tau_{rz} = 0, \quad \tau_{\theta z} = 0 \end{cases}$

WHERE,
 $M_1 = \frac{\partial}{\partial r} (2\beta - \frac{3}{2}\epsilon)$ (TO HAVE THESE IN TERMS OF ϕ & χ , WE REPLACE $\nabla^2 \chi$ BY $-\psi_0$)
 $N_1 = -\frac{[\alpha(3\lambda_0 + 2\mu_0)]}{(3\alpha + 2\beta) \frac{\partial}{\partial r} (\frac{1}{r^2})} \left[\frac{-(3\alpha + 4\beta) \frac{\partial^2 \phi}{\partial r^2} + 2\beta [(3\alpha + 2\beta) \frac{\partial}{\partial r} (\frac{1}{r^2}) + \frac{1}{r} \frac{\partial^2 \chi}{\partial r^2}] \right] \nabla^2 \phi$

9) TELEGRAPH EQUATIONS FOR THE SCALAR & VECTOR POTENTIALS:

i) $[2\beta + \alpha \frac{\partial}{\partial r} (\frac{1}{r^2}) + \frac{3}{2}\epsilon] \nabla^2 \phi = \frac{1}{\epsilon} \{ [4\beta + 3\alpha + \frac{3}{2}\epsilon] \frac{\partial^2 \phi}{\partial r^2} + 2\beta (3\alpha + 2\beta) \frac{\partial \phi}{\partial r} \}$
 ii) $\nabla^2 \psi = \frac{1}{\epsilon} (2\beta + \frac{3}{2}\epsilon) \frac{\partial \psi}{\partial r}$ AND $\nabla \cdot \vec{\psi} = 0$

FOR $c_s^2 \ll c^2$ (i.e. $\mu_0 \ll \lambda_0$ OR $\nu \ll 1/2$, i.e. RUBBER) EQ. (2) IS:
 i) $\nabla^2 \phi = \frac{1}{\epsilon} \{ \frac{\partial^2 \phi}{\partial r^2} + (3\alpha + 2\beta) \frac{\partial \phi}{\partial r} \}$

10) TELEGRAPH EQUATIONS FOR THE INDEPENDENT SCALAR POTENTIALS:

THE EQ. FOR ϕ IS THE SAME AS ABOVE IN 9). THE EQ. FOR χ IS:
 $-\nabla^2 \{ \nabla^2 \chi - \frac{1}{\epsilon} (2\beta + \frac{3}{2}\epsilon) \frac{\partial \chi}{\partial r} \} = 0$ OR IN TERMS OF ψ_0 ,
 $\nabla^2 \psi_0 = \frac{1}{\epsilon} (2\beta + \frac{3}{2}\epsilon) \frac{\partial \psi_0}{\partial r}$

TELEGRAPH EQUATION (1) AND (2) ARE COORDINATE INDEPENDENT FOR GENERAL CYLINDRICAL COORDINATES BUT NOW WITHOUT THE λ DEPENDENCE. NOW THAT THE SOLENOIDAL SOLUTION OF $(1 + N \frac{\partial}{\partial r}) \nabla^2 \psi = \frac{1}{\epsilon} \frac{\partial \psi}{\partial r}$ IS, $\psi = -\frac{1}{\epsilon} \nabla^2 \chi$ (A), THE TELEGRAPH EQUATION, I.E. $(1 + N \frac{\partial}{\partial r}) \nabla^2 \chi = \frac{1}{\epsilon} \frac{\partial \chi}{\partial r}$. NOTE THAT ACT ON A λ -INDEPENDENT FUNCTION, THEY ARE ALSO λ -INDEPENDENT IN THIS CASE. $\psi_0 = 0$, $\psi_0 = -\nabla^2 \chi(r, \theta, t)$ (B). SO THE VECTOR POTENTIAL HAS ONLY ONE COMPONENT. SUBSTITUTING SLN. (A) INTO $\vec{u} = \nabla \phi + \nabla \times \vec{\psi}$ YIELDS THE DISPLACEMENT-DEPENDENT EQ. WE REPEAT THAT ALL THIS HOLDS IN THE ABSENCE OF DAMPING WHEN THE EQUATION, AS IT OCCURS IN ELASTICITY, PROVIDED THAT χ THEN SATISFIES A WAVE EQUATION.

10) TELEGRAPH-TYPE EQS. 10) THEN THE FIELD-EQS 7) ARE AUTOMATICALLY SATISFIED. THIS TECHNIQUE, WHICH IS BASIC TO SET-UP AND SOLVE, WAS DEVELOPPED FOR ALL THESE CASES AND VISCOELASTIC MODELS. COULOMB (i.e. $\nabla \cdot \vec{\psi} = 0$) RATHER THAN A LORENTZ (i.e. $\nabla \cdot \vec{\psi} = \frac{1}{\epsilon} \frac{\partial \chi}{\partial r}$) GAUGE. KELVIN-VOIGT MODEL, AND/OR $\alpha = 0$, $\beta = 0$ FOR THE MAXWELL MODEL). THE PLANE-STRAIN CASES BY λ_0 DEFINED TO BE: $\lambda_0 = 2\lambda_0 \mu_0 / (\lambda_0 + 2\mu_0)$. IF $\lambda_0 = 0$, THEN THERE IS ONLY r -DEPENDENCE (i.e. $u_r = u_r(r, t)$ & $\phi = \phi(r, t)$). $u_\theta = 0$ (KELVIN-VOIGT) OR $3\alpha + 2\beta = 0$ (MAXWELL) SO FORMULAS SIMPLIFY A BIT.

2. $\epsilon_{rr} = \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2}$, $2\epsilon_{\theta\theta} = \frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2}$

3). EQS. 4) INTO EQS. 3).

7) FIELD EQUATIONS:

i) $[\mu_0 + \mu_0 \frac{\partial}{\partial r}] \nabla^2 \vec{u} + [\lambda_0 + \mu_0 + (\lambda_0 + \mu_0) \frac{\partial}{\partial r}] \nabla (\nabla \cdot \vec{u}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$ OR IF

$\begin{cases} [\mu_0 + \mu_0 \frac{\partial}{\partial r}] [\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r} \frac{\partial u_\theta}{\partial \theta}] + [\lambda_0 + \mu_0 + (\lambda_0 + \mu_0) \frac{\partial}{\partial r}] \frac{\partial \theta}{\partial r} = \rho \frac{\partial^2 u_r}{\partial t^2} \\ [\mu_0 + \mu_0 \frac{\partial}{\partial r}] [\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r} \frac{\partial u_r}{\partial \theta}] + [\lambda_0 + \mu_0 + (\lambda_0 + \mu_0) \frac{\partial}{\partial r}] \frac{1}{r} \frac{\partial \theta}{\partial \theta} = \rho \frac{\partial^2 u_\theta}{\partial t^2} \\ [\mu_0 + \mu_0 \frac{\partial}{\partial r}] [\nabla^2 u_r - \frac{u_r}{r^2} + \frac{2}{r} \frac{\partial u_\theta}{\partial \theta}] + [\lambda_0 + \mu_0 + (\lambda_0 + \mu_0) \frac{\partial}{\partial r}] \frac{1}{r} \frac{\partial \theta}{\partial r} = \rho \frac{\partial^2 \theta}{\partial t^2} \end{cases}$

ii) $[\lambda_0 + 2\mu_0 + (\lambda_0 + 2\mu_0) \frac{\partial}{\partial r}] \nabla (\nabla \cdot \vec{u}) - [\mu_0 + \mu_0 \frac{\partial}{\partial r}] \nabla \times (\nabla \times \vec{u}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$

THE DILATATION Δ IS GIVEN IN 5) ABOVE AND, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$\begin{cases} \tau_{rr} = [\lambda_0 + \lambda_0 \frac{\partial}{\partial r}] \nabla^2 \phi + 2[\mu_0 + \mu_0 \frac{\partial}{\partial r}] [\frac{\partial^2 \phi}{\partial r^2} (\phi + \frac{\partial}{\partial r} (r \psi)) - \frac{\partial}{\partial r} (r \nabla^2 \psi)] \\ \tau_{\theta\theta} = [\lambda_0 + \lambda_0 \frac{\partial}{\partial r}] \nabla^2 \phi + 2[\mu_0 + \mu_0 \frac{\partial}{\partial r}] [\frac{1}{r} \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2}) (\phi + \frac{\partial}{\partial r} (r \psi)) - \nabla^2 \psi - \frac{1}{r} \frac{\partial \psi}{\partial r}] \\ \tau_{r\theta} = [\lambda_0 + \lambda_0 \frac{\partial}{\partial r}] \nabla^2 \phi + 2[\mu_0 + \mu_0 \frac{\partial}{\partial r}] [\frac{1}{r} \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2}) (\phi + \frac{\partial}{\partial r} (r \psi)) + \frac{1}{r} \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2}) (\phi + \frac{\partial}{\partial r} (r \psi)) \\ \tau_{\theta r} = [\mu_0 + \mu_0 \frac{\partial}{\partial r}] [\frac{2}{r} \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2}) (\phi + \frac{\partial}{\partial r} (r \psi)) + (\frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2}) (\phi + \frac{\partial}{\partial r} (r \psi)) \\ \tau_{rz} = [\mu_0 + \mu_0 \frac{\partial}{\partial r}] [\frac{2}{r} \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2}) (\phi + \frac{\partial}{\partial r} (r \psi)) - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} (\nabla^2 \psi) + \frac{\partial}{\partial r} (\frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{\partial \psi}{\partial \theta}) \\ \tau_{\theta z} = [\mu_0 + \mu_0 \frac{\partial}{\partial r}] [2 \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2}) (\phi + \frac{\partial}{\partial r} (r \psi)) - \frac{\partial}{\partial r} (\nabla^2 \psi) - \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta^2} (\frac{\partial \phi}{\partial r} - \frac{1}{r} \frac{\partial \psi}{\partial \theta})] \end{cases}$

9) TELEGRAPH EQUATIONS FOR THE SCALAR AND VECTOR POTENTIALS:

i) $(1 + M \frac{\partial}{\partial r}) \nabla^2 \phi = \frac{1}{\epsilon} \frac{\partial \phi}{\partial r}$ ii) $(1 + N \frac{\partial}{\partial r}) \nabla \times (\nabla \times \vec{\psi}) +$
 $M = \frac{\lambda_0 + 2\mu_0}{\lambda_0 + 2\mu_0}$, $N = \frac{\mu_0}{\mu_0}$, $c_s^2 = \frac{\lambda_0 + 2\mu_0}{\rho}$, $c_t^2 = \frac{\mu_0}{\rho}$

THESE EQS. ARE OBTAINED BY SUBSTITUTING $\vec{u} = \nabla \phi + \nabla \times \vec{\psi}$ INTO OF THE FIELD EQS. 7). EQ. (2) CAN ALSO BE WRITTEN AS
 (i) $(1 + N \frac{\partial}{\partial r}) \nabla^2 \vec{\psi} = \frac{1}{\epsilon} \frac{\partial \vec{\psi}}{\partial r}$ AND $\nabla \cdot \vec{\psi} = 0$

10) TELEGRAPH EQUATIONS FOR THE INDEPENDENT SCALAR POTENTIALS:

i) $(1 + M \frac{\partial}{\partial r}) \nabla^2 \phi = \frac{1}{\epsilon} \frac{\partial \phi}{\partial r}$ ii) $(1 + N \frac{\partial}{\partial r}) \nabla^2 \psi = \frac{1}{\epsilon} \frac{\partial \psi}{\partial r}$

WHERE, $M = \frac{\lambda_0 + 2\mu_0}{\lambda_0 + 2\mu_0}$, $N = \frac{\mu_0}{\mu_0}$, $c_s^2 = \frac{\lambda_0 + 2\mu_0}{\rho}$, $c_t^2 = \frac{\mu_0}{\rho}$

11) SOLENOIDAL SOLUTION OF THE VECTOR TELEGRAPH EQ. N.B. IT CAN BE SHOWN THAT THE SOLUTION OF EQS. 9) (2) ABOVE IS GIVEN BY $\vec{\psi} = \nabla \times (\vec{A} \psi) + \nabla \times (\vec{B} \chi) = (\nabla \psi) \times \vec{r} + \nabla \times (\vec{B} \chi)$ PROVIDED THAT THE TWO NEW SCALAR FUNCTIONS ψ & χ SATISFY THE SCALAR TELEGRAPH EQS (i.e. $(1 + M \frac{\partial}{\partial r}) \nabla^2 \psi = \frac{1}{\epsilon} \frac{\partial \psi}{\partial r}$ AND $(1 + N \frac{\partial}{\partial r}) \nabla^2 \chi = \frac{1}{\epsilon} \frac{\partial \chi}{\partial r}$) (NOTE THAT ψ IS NOT $|\vec{\psi}|$). OPENING UP THE SOLUTION (A) YIELDS $\vec{\psi} = -\frac{1}{r} \frac{\partial \psi}{\partial r} \vec{r} - \frac{1}{r} \frac{\partial \chi}{\partial \theta} \vec{\theta} - \frac{1}{r} \frac{\partial \chi}{\partial r} \vec{r} \times \vec{\theta}$, $\psi_0 = \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial \chi}{\partial \theta} + \frac{1}{r} \frac{\partial \chi}{\partial r}$ WHICH RELATE THE 3 SPHERICAL COMPONENTS OF THE VECTOR POTENTIAL SOLENOIDAL BY THE GAUGE CONDITION $\nabla \cdot \vec{\psi} = 0$ TO THE 2 SCALAR SOLUTION (A) IS VALID IN SPHERICALS ONLY, AND WHEN IT IS SUBSTITUTED INTO THE VECTOR TELEGRAPH EQ. THE RESULT IS THE DISPLACEMENT-INDEPENDENT POTENTIAL EQUATION. AUXILIARY RELATIONS NEEDED TO PROVE THIS ARE: $\nabla \cdot (\vec{\psi} \times \vec{r}) = [\nabla \cdot (\vec{\psi} \times \vec{r})] \cdot \vec{r}$, $\nabla \cdot (\vec{\psi} \times \vec{r}) = [\nabla \cdot (\vec{\psi} \times \vec{r})] \cdot \vec{r}$. IN THE ABSENCE OF VISCOUSITY (I.E. $\lambda_0 = 0$) ABOVE IS THE SOLENOIDAL SOLUTION OF THE VECTOR WAVE EQ. PROVIDED THAT THEN ψ AND χ SATISFY 2 SCALAR WAVE-EQS.

12) REMARKS:

i) REMARKS a), b), c) OF THE CYLINDRICAL CASE TO THE LEFT ALSO APPLY. NOTE AGAIN THAT IN SPITE OF THE APPARENT GENERALITY OF THE ABOVE WITH ITS VECTOR OPERATORS, IT IS ONLY VALID IN SPHERICALS. FURTHER THAT THE TWO PARTS (i.e. $\nabla \cdot (\vec{\psi} \times \vec{r})$ AND $\nabla \cdot (\vec{\psi} \times \vec{r})$ THEY ARE USUALLY PERPENDICULAR) EVEN WHEN $\psi = \chi$.

ii) THERE IS NO WAY TO HAVE A PLANE-STRAIN OR PLANE-STRESS CASE. THE CLOSEST WE CAN GET TO THEM IS THE AXIALLY SYMMETRIC CASE.

$$2\epsilon_{\phi\phi} = 0, \quad 2\epsilon_{\phi r} = 0,$$

$$2\epsilon_{rr} = 2\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}(\varphi + \frac{\partial}{\partial r}(r\psi))\right) - \frac{\partial}{\partial\theta}(\nabla^2\psi)\right] = \frac{2}{r}\frac{\partial^2\varphi}{\partial\theta^2} - \frac{2}{r}\frac{\partial\varphi}{\partial\theta} +$$

$$+ \left(\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial}{\partial\theta}\right)\frac{\partial\psi}{\partial\theta} + \frac{2}{r^2}\frac{\partial\psi}{\partial\theta}.$$

$$[\rho_0 + \mu_x + (\lambda_x + \mu_y) \frac{\partial}{\partial x}] \nabla \cdot (\vec{\nabla} \cdot \vec{u}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2} \quad \text{OR IN COMPONENT-FORM,}$$

$$-\frac{2\mu_0 \cos \theta}{r \sin \theta} \frac{\partial u}{\partial \theta} + [\lambda_e + \mu_e + (\lambda_r + \mu_r) \frac{r}{\theta}] \frac{1}{r} \frac{\partial}{\partial \theta} = \rho \frac{\partial^2 u}{\partial t^2}$$

IN (5) ABOVE AND, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$

POTENTIAL RELATIONS:

$$\begin{aligned}
 & + \mu \frac{\partial}{\partial t} \left[\left(\frac{\partial^2}{\partial t^2} \left(\psi + \frac{\partial}{\partial r} (r\psi) \right) - \frac{\partial}{\partial r} (r \nabla^2 \psi) \right) \right] \\
 & + \mu \frac{\partial}{\partial t} \left[\left(\left(\frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial^2}{\partial \theta^2} \right) \left(\psi + \frac{\partial}{\partial r} (r\psi) \right) - \nabla^2 \psi - \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\nabla^2 \chi) \right) \right) \right] \\
 & + \mu \frac{\partial}{\partial t} \left[\left(\left(\frac{1}{r} \frac{\partial}{\partial r} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \left(\psi + \frac{\partial}{\partial r} (r\psi) \right) - \nabla^2 \psi + \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\nabla^2 \chi) \right) \right) \right. \\
 & \left. \frac{\partial}{\partial \phi} \left(\psi + \frac{\partial}{\partial r} (r\psi) \right) + \left(\frac{1}{r} \frac{\partial^2}{\partial \theta^2} - \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} - \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) (\nabla^2 \chi) \right] \\
 & \left. \frac{\partial}{\partial \phi} \left(\psi + \frac{\partial}{\partial r} (r\psi) \right) \right\} - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\nabla^2 \psi) + \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) (\nabla^2 \chi) \Big] \\
 & \left(\psi + \frac{\partial}{\partial r} (r\psi) \right) \Big\} - \frac{\partial}{\partial r} (\nabla^2 \psi) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) (\nabla^2 \chi) \Big]
 \end{aligned}$$

[THESE ARE EQS.
6) SUBSTITUTED
INTO EQ 6. 2)

(THESE ARE EQS.
6) SUBSTITUTED
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FOR THE SCALAR AND VECTOR POTENTIALS ϕ AND $\vec{\psi}$:

$$(c) \quad (1 + N \frac{\partial}{\partial x}) \nabla \cdot (\nabla \times \vec{\psi}) + \frac{1}{q} \frac{\partial^2 \vec{\psi}}{\partial x^2} = 0 \quad \text{WHERE,}$$

BY SUBSTITUTING $\vec{u} = \vec{v}\phi + \vec{v}\cdot\vec{\nabla}\phi$ INTO THE ALTERNATIVE FORM
EQ. (1) CAN ALSO BE WRITTEN AS FOLLOWS,
 $\vec{v}\cdot\vec{\nabla}\phi = \frac{1}{2} \frac{d\phi}{dt}$ AND $\vec{v}\cdot\vec{\nabla} = 0$

FOR THE INDEPENDENT SCALAR POTENTIALS ψ, ψ, χ :

$$N = \frac{H}{L}, \quad G^1 = \frac{2\mu + 2H\mu}{P}, \quad G^2 = \frac{H\mu}{L}, \quad \text{L.V. AS IN EQ. (7).}$$

OF THE VECTOR TELEGRAPH. EQ. V GENERAL SPHERICAL COORDS

SOLUTION OF EQS. 9) & 10) ABOVE IS GIVEN BY,

$$P = \frac{1}{\sqrt{\pi}} \left[\frac{1}{2} \left(\frac{1}{w} + \frac{1}{v} \right) + \frac{1}{2} \left(\frac{1}{w} - \frac{1}{v} \right) \right] = \frac{1}{\sqrt{\pi}} \left[\frac{1}{2} \left(\frac{1}{w} + \frac{1}{v} \right) + \frac{1}{2} \left(\frac{1}{w} - \frac{1}{v} \right) \right] = \frac{1}{\sqrt{\pi}} \left[\frac{1}{2} \left(\frac{1}{w} + \frac{1}{v} \right) + \frac{1}{2} \left(\frac{1}{w} - \frac{1}{v} \right) \right] \quad (A)$$

NEW SCALAR FUNCTIONS $\psi(r, \theta, 0, 0)$ & $\chi(r, 0, 0, 0)$ SATISFY
(i.e. $(1+n\frac{r}{2})\nabla^2\psi = \frac{1}{2}\frac{\partial^2\psi}{\partial r^2}$ AND $(1+n\frac{r}{2})\nabla^2\chi = \frac{1}{2}\frac{\partial^2\chi}{\partial r^2}$.)

$$\text{radius } \frac{1}{2} \sqrt{2}, \quad y = \frac{1}{\sin \theta} \frac{dy}{d\theta} + \frac{1}{\sin^2 \theta} \frac{d^2 y}{d\theta^2}, \quad y = \frac{1}{\sin \theta} \cdot \frac{1}{\sin \theta} \frac{dy}{d\theta} + \frac{1}{\sin^2 \theta} \frac{d^2 y}{d\theta^2}$$

PHYSICAL COMPONENTS OF THE VECTOR POTENTIAL $\vec{\psi}$ (WHICH IS ALSO THE CONDITION $\nabla \cdot \vec{\psi} = 0$) TO THE 2 SCALARS ψ AND χ .

PHERICALS ONLY, AND WHEN IT IS SUBSTITUTED INTO $\psi = \psi_0 + \psi_1$
 PLACEMENT - INDEPENDENT POTENTIAL RELATIONS IN EQS. 4) ABOVE

TO PROVE THIS ARE: $\nabla_{\alpha}((\nabla\psi)_{\alpha}^{\beta}) = \nabla\psi - \nabla\psi - \nabla\psi = \nabla(\nabla\psi \cdot \mathbf{f})$
 $\nabla^2 \mathbf{f}$. IN THE ABSENCE OF VISCOSITY (i.e., $\mu = \eta = 0$, i.e., ELASTICITY
 THE DIRECTION OF THE VECTOR WAS: $\nabla^2 \mathbf{f} = \nabla(\nabla\psi \cdot \mathbf{f}) = 0$

AND χ SATISFY 2 SCALAR WAVE-EQS. (i.e. $(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \chi = 0 \dots$)

THE CYLINDRICAL CASES TO THE LEFT ALSO HOLD HERE.
 (NOTE OF THE APPARENT GENERALITY OF THE SOLENOIDAL SOLUTION 11)

FOR OPERATORS, IT IS ONLY VALID IN SPHERICAL COORDINATES. NOTE
PARTS (i.e. $\nabla_0(FV)$ AND $\nabla_0[\nabla_0(F\lambda)]$) ARE DIFFERENT (i.e.

PERPENDICULAR) EVEN WHEN $\psi \neq \lambda$.

THE A PLANE-STRAIN OR PLANE-STRESS SITUATION IN SPHERICAL CASE
SET TO THEM IS THE AXIALLY SYMMETRIC CASE (b) TO THE RIGHT.

7) FIELD EQUATIONS :

$$\begin{aligned} i) \quad & [\mu_o + \mu_o \frac{\partial}{\partial t}] [\nabla_o^2 u - \frac{2u}{r^2} - \frac{2}{r} \frac{\partial u}{\partial r} - \frac{2u \cot \theta}{r^2}] + [\lambda_o + \mu_o + (\lambda_o + \mu_o) \frac{\partial}{\partial t}] \frac{\partial u}{\partial t} = \\ & - \rho \frac{\partial^2 u}{\partial t^2} \\ & [\mu_o + \mu_o \frac{\partial}{\partial t}] [\nabla_o^2 u_o + \frac{2}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2 \sin^2 \theta}] + [\lambda_o + \mu_o + (\lambda_o + \mu_o) \frac{\partial}{\partial t}] \frac{1}{r} \frac{\partial u}{\partial \theta} = \rho \frac{\partial^2 u}{\partial t^2} \\ \text{WHERE,} \quad & \nabla_o^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} \\ & \Delta = \frac{\partial u}{\partial r} + \frac{2u}{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{u}{r} \cot \theta \end{aligned}$$

FORM 44) WILL NOT BE GIVEN HERE.

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$$\begin{aligned} \mathcal{L}_{rr} &= (\lambda_0 + \lambda_0 \frac{\partial}{\partial t}) \nabla^2 \varphi + 2(\mu_0 + \mu_0 \frac{\partial}{\partial t}) \left[\frac{\partial^2}{\partial r^2} (\varphi + \frac{\partial}{\partial r} (r\psi)) - \frac{\partial}{\partial r} (r \nabla_0^2 \varphi) \right] \\ \mathcal{L}_{\theta\theta} &= (\lambda_0 + \lambda_0 \frac{\partial}{\partial t}) \nabla^2 \varphi + 2(\mu_0 + \mu_0 \frac{\partial}{\partial t}) \left[\left(\frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) (\varphi + \frac{\partial}{\partial r} (r\psi)) - \nabla_0^2 \varphi \right] \\ \mathcal{L}_{\varphi\varphi} &= (\lambda_0 + \lambda_0 \frac{\partial}{\partial t}) \nabla^2 \varphi + 2(\mu_0 + \mu_0 \frac{\partial}{\partial t}) \left(\frac{1}{r} \frac{\partial}{\partial r} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2} \right) (\varphi + \frac{\partial}{\partial r} (r\psi)) - \nabla_0^2 \varphi \\ \mathcal{L}_{\theta\varphi} &= 0, \quad \mathcal{L}_{r\varphi} = 0, \end{aligned}$$

$$\epsilon_0 = (\mu_e + \mu_p \frac{g}{2}) \left[2 \frac{g}{3} \left\{ \frac{1}{2} \frac{g}{3} (\phi + \frac{g}{2} (r\psi)) \right\} - \frac{g}{3} (\nabla^2 \psi) \right].$$

ALTHOUGH $\psi_e = -\frac{g}{2}$ IT IS NOT CONVENIENT TO EXPRESS NEITHER THESE RELATIONS NOR G-4) IN TERMS OF ϕ AND ψ .
(THESE ARE EQS. 6) SUBSTITUTED INTO EQS. 21) ($\Delta = \nabla^2 \phi$).

W. TELEGRAPH EQS. FOR SCALAR & VECTOR POTENTIALS p. 5

(1) $(1 + M \frac{\partial^2}{\partial t^2}) \nabla_{\perp}^2 \psi = \frac{1}{\psi} \frac{\partial^2 \psi}{\partial t^2}$, (ii) $(1 + N \frac{\partial^2}{\partial t^2}) \nabla_{\parallel} (\psi \cdot \nabla) + \frac{1}{\psi} \frac{\partial^2 \psi}{\partial t^2} = 0$
 WHERE M, N, ψ, ψ ARE GIVEN TO THE LEFT 4 EQS. (F-9.).
 EQ. (2) CAN ALSO BE WRITTEN IN THE FORM:
 (a) $(1 + N \frac{\partial^2}{\partial t^2}) \nabla_{\parallel}^2 \psi = \frac{1}{\psi} \frac{\partial^2 \psi}{\partial t^2}$ & $\psi \cdot \nabla = 0$
 ψ^2 IS GIVEN IN ??).

201 TELEGRAPH EQS. FOR THE INDEPENDENT SCALAR POTENTIALS:

(1) $(1 + M \frac{\partial}{\partial t}) \nabla^2 \psi = \frac{1}{\epsilon} \frac{\partial^2 \psi}{\partial t^2}$ (2) $(1 + N \frac{\partial}{\partial t}) \nabla^2 \psi = \frac{1}{\epsilon} \frac{\partial^2 \psi}{\partial t^2}$
 WHERE M, N, ϵ , ψ , ∇^2 ARE AS BEFORE. SINCE $\psi = -\frac{eV}{8\pi}$
 THE TELEGRAPH-TYPE EQ. FOR ψ IS:
 (3) $(1 + N \frac{\partial}{\partial t}) [\nabla^2 \psi - \frac{\psi}{\epsilon \tau^2}] = \frac{1}{\epsilon} \frac{\partial^2 \psi}{\partial t^2}$
 WHICH IS NOT AS CONVENIENT AS (2).

11) DIFFERENTIAL EQUATION OF VECTOR TELEGRAPH EQUATION:

THE SOLUTION OF EQS. 9 & 10 IN SPHERICAL COORDS. WITH AXIAL SYMMETRY IS NOW GIVEN BY: $\vec{U} = \vec{U}_r(r, \psi) = (U_r, U_\psi, 0)$ WHICH IN COMPONENT FORM IS: $U_r = 0, U_\psi = 0, U_\phi = -\frac{1}{r} \frac{\partial \psi}{\partial \psi}$. (NOTE AGAIN THAT ψ IS NOT $|\vec{U}|$.) PROVIDED THAT $\psi(r, \psi) = \frac{1}{r} \frac{\partial \psi}{\partial \psi}$ SATISFIES A SCALAR TELEGRAPH EQ. $(1 - N^2 \frac{\partial^2}{\partial r^2}) \nabla^2 \psi = \frac{1}{r} \frac{\partial \psi}{\partial \psi}$. SUBSTITUTING SOLUTION (A) INTO $\vec{U} = \vec{U}_r(r, \psi) = (U_r, U_\psi, 0)$ YIELDS $\vec{U} = \vec{U}_r[\psi + \frac{1}{2} (r \psi)] - \vec{r} \nabla^2 \psi$ AS GIVEN ABOVE IN EQS. 4).

AGAIN, IN THE ABSENCE OF VISCOSITY (i.e., $M=N=0$, i.e., ELASTICITY) EQ. (A) IS THE SOLENOIDAL SOLUTION OF THE VECTOR WAVE-EQ. $(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \vec{v} = 0$ IN SPHERICAL COORDS. WITH AXIAL SYMMETRY, PROVIDED THAT THE SCALAR FUNCTION $\chi(x, \theta, r, t)$ SATISFIES A SCALAR WAVE-EQUATION. (i.e., $[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}] \chi = 0$)

NOTE THAT EQ. 10 (A) FOR ψ IS MORE COMPLICATED THAN
10 (1), HENCE, EQS. 4) & 8) ARE BEST LEFT IN TERMS OF φ & ψ .

(2) 26 MAR 64

2) REMARKS a), b), c) OF THE CYLINDRICAL CASES TO THE LEFT ALSO HOLD HERE.
 22) THE MAXWELL MODEL IN SPHERICAL COORDS. WILL BE LEFT AS AN EXERCISE FOR THE READER.
 222) THIS ENTIRE CHART, VALID FOR ARBITRARY TIME-DEPENDENCE, WILL BE PARTICULARIZED FOR HARMONIC TIME-DEPENDENCE OF THE FORM $e^{-i\omega t}$ IN A COMPANION CHART. (G. GAUNAUD, '75)

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TABLE 2: EQUATIONS OF LINEAR DYNAMIC VISCOELASTICITY IN CYLINDRICAL AND SPHERICAL COORDINATES

1) RELATION BETWEEN THE STRESS ϵ_{kk} & DILATION Δ :	2) RELATION BETWEEN THE STRESS ϵ_{kk} & DILATION Δ :	3) STRAIN-DISPLACEMENT RELATIONS:
$\epsilon_{kk} = [3\lambda_e + 2\mu_e - i\omega(3\lambda_v + 2\mu_v)] \epsilon_{kk}$ $\epsilon_{kk} = \text{BULK STRESS} \quad \epsilon_{kk} = \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz} = \Delta = \text{DILATION}$	$\epsilon_{kk} = \frac{3\lambda_e + 2\mu_e}{\omega^2 + (3\alpha + 2\beta)^2} [\omega^2 - i\omega(3\alpha + 2\beta)] \epsilon_{kk}$ $\epsilon_{kk} = \text{BULK STRESS} \quad \epsilon_{kk} = \text{DILATION} \quad \alpha, \beta = \text{VISCOUS CONSTANTS}$	$\epsilon_{ij} = (\lambda_e - i\omega\lambda_v) \delta_{ij} \epsilon_{kk} + 2(\mu_e - i\omega\mu_v) \epsilon_{ij}$ <p>HERE, $\lambda_e, \mu_e = \text{ELASTIC (LAME) CONSTANTS}$ $\lambda_v, \mu_v = \text{VISCOUS CONSTANTS}$</p>
4) DISPLACEMENT - INDEPENDENT POTENTIAL RELATIONS:	5) DISPLACEMENT - INDEPENDENT POTENTIAL RELATIONS:	6) DISPLACEMENT - INDEPENDENT POTENTIAL RELATIONS:
$u_i = \nabla(\phi + \frac{\partial \psi}{\partial x_i}) + \kappa_i^2 [\psi \delta_{ij} + (\nabla \chi)_j \delta_{ij}]$ <p>OR IN COMPONENT FORM,</p> $u_r = \frac{\partial}{\partial r}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_r^2 \frac{\partial \chi}{\partial r} \quad u_\theta = \frac{1}{r} \frac{\partial}{\partial \theta}(\phi + \frac{\partial \psi}{\partial r}) - \kappa_\theta^2 \frac{\partial \chi}{\partial r} \quad u_z = \frac{\partial}{\partial z}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_z^2 \psi$ <p>HERE, $\phi = \phi(r, \theta, z) e^{-i\omega t}$, $\psi = \psi(r, \theta, z) e^{-i\omega t}$, $\chi = \chi(r, \theta, z) e^{-i\omega t}$, $\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$</p>	$\epsilon_{ij} = (\frac{\omega^2 + i\omega 2\beta}{\omega^2 + 4\beta^2}) [\lambda_e \delta_{ij} \Delta + 2\mu_e \epsilon_{ij}] + N_i$ <p>WHERE N_i IS GIVEN IN 2), $2\epsilon_{ij} = \epsilon_{ij} + \epsilon_{ji}$ ARE GIVEN IN 3) AND $\Delta = \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$</p>	$\epsilon_{rr} = \frac{\partial^2}{\partial r^2}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_r^2 \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})$ $\epsilon_{\theta\theta} = (\frac{1}{r} \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2})(\phi + \frac{\partial \psi}{\partial r}) - \kappa_\theta^2 \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})$ $\epsilon_{zz} = \frac{\partial^2}{\partial z^2}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_z^2 \frac{\partial \psi}{\partial r}$ $2\epsilon_{r\theta} = 2 \frac{\partial^2}{\partial r \partial \theta}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_\theta^2 \frac{\partial \psi}{\partial r} - \kappa_r^2 \frac{\partial^2 \chi}{\partial r^2}$ $2\epsilon_{rz} = 2 \frac{\partial^2}{\partial r \partial z}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_z^2 \frac{\partial \psi}{\partial r} - \kappa_r^2 \frac{\partial^2 \chi}{\partial r^2}$ $2\epsilon_{\theta z} = 2 \frac{\partial^2}{\partial \theta \partial z}(\phi + \frac{\partial \psi}{\partial r}) - \kappa_z^2 (\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial^2 \chi}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2})$
7) FIELD EQUATIONS:	7) FIELD EQUATIONS:	7) FIELD EQUATIONS:
$\nabla^2 \bar{u} + [1 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}] \nabla(\nabla \cdot \bar{u}) + \kappa_i^2 \bar{u} = 0$ <p>WHERE,</p> $\kappa_i^2 = \rho \omega^2 / (\mu_e - i\omega\mu_v)$ <p>THIS EQ. CAN BE OPENED UP BY MEANS OF $\nabla^2 \bar{u} = [\nabla^2 u_r - \frac{1}{r} \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})] \hat{e}_r + [\nabla^2 u_\theta - \frac{1}{r} \frac{\partial}{\partial \theta}(\frac{1}{r} \frac{\partial \psi}{\partial r})] \hat{e}_\theta + [\nabla^2 u_z - \frac{1}{r} \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})] \hat{e}_z$</p> <p>AND $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$</p> <p>THE GRADIENT OPERATOR & THE DILATION Δ ARE GIVEN ABOVE.</p>	$\nabla^2 \bar{u} + [1 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}] \nabla(\nabla \cdot \bar{u}) + \kappa_i^2 \bar{u} = 0$ <p>OR IN COMPONENT FORM,</p> $[\nabla^2 u_r - \frac{1}{r} \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})] + \kappa_r^2 u_r = 0$ $[\nabla^2 u_\theta - \frac{1}{r} \frac{\partial}{\partial \theta}(\frac{1}{r} \frac{\partial \psi}{\partial r})] + \kappa_\theta^2 u_\theta = 0$ $[\nabla^2 u_z - \frac{1}{r} \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})] + \kappa_z^2 u_z = 0$ <p>WHERE, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$</p>	$\nabla^2 \bar{u} + [1 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}] \nabla(\nabla \cdot \bar{u}) + \kappa_i^2 \bar{u} = 0$ <p>OR IN COMPONENT FORM,</p> $[\nabla^2 u_r - \frac{1}{r} \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})] + \kappa_r^2 u_r = 0$ $[\nabla^2 u_\theta - \frac{1}{r} \frac{\partial}{\partial \theta}(\frac{1}{r} \frac{\partial \psi}{\partial r})] + \kappa_\theta^2 u_\theta = 0$ $[\nabla^2 u_z - \frac{1}{r} \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})] + \kappa_z^2 u_z = 0$ <p>WHERE, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$</p>
8) STRAIN - INDEPENDENT POTENTIAL RELATIONS:	8) STRAIN - INDEPENDENT POTENTIAL RELATIONS:	8) STRAIN - INDEPENDENT POTENTIAL RELATIONS:
$\epsilon_{rr} = (\lambda_e - i\omega\lambda_v) \nabla^2 \phi + 2(\mu_e - i\omega\mu_v) [\frac{\partial^2}{\partial r^2}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_r^2 \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})]$ $\epsilon_{\theta\theta} = (\lambda_e - i\omega\lambda_v) \nabla^2 \phi + 2(\mu_e - i\omega\mu_v) [\frac{1}{r} \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}](\phi + \frac{\partial \psi}{\partial r}) - \kappa_\theta^2 \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})]$ $\epsilon_{zz} = (\lambda_e - i\omega\lambda_v) \nabla^2 \phi + 2(\mu_e - i\omega\mu_v) [\frac{\partial^2}{\partial z^2}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_z^2 \frac{\partial \psi}{\partial r}]$ $\epsilon_{r\theta} = (\mu_e - i\omega\mu_v) [\frac{\partial^2}{\partial r \partial \theta}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_\theta^2 \frac{\partial \psi}{\partial r} - \kappa_r^2 \frac{\partial^2 \chi}{\partial r^2}]$ $\epsilon_{rz} = (\mu_e - i\omega\mu_v) [2 \frac{\partial^2}{\partial r \partial z}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_z^2 \frac{\partial \psi}{\partial r} - \kappa_r^2 \frac{\partial^2 \chi}{\partial r^2}]$ $\epsilon_{\theta z} = (\mu_e - i\omega\mu_v) [2 \frac{\partial^2}{\partial \theta \partial z}(\phi + \frac{\partial \psi}{\partial r}) - \kappa_z^2 (\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial^2 \chi}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2})]$ <p>THESE ARE EQS 5) WITH $\Delta = \nabla^2 \phi + \kappa_i^2 \phi$ AND ϵ_{ij} AS GIVEN IN 6).</p>	$\epsilon_{rr} = M_1 (\lambda_e \nabla^2 \phi + 2\mu_e [\frac{\partial^2}{\partial r^2}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_r^2 \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})]) + N_1$ $\epsilon_{\theta\theta} = M_1 (\lambda_e \nabla^2 \phi + 2\mu_e [\frac{1}{r} \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}](\phi + \frac{\partial \psi}{\partial r}) - \kappa_\theta^2 \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})) + N_1$ $\epsilon_{zz} = M_1 (\lambda_e \nabla^2 \phi + 2\mu_e [\frac{\partial^2}{\partial z^2}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_z^2 \frac{\partial \psi}{\partial r}]) + N_1$ $\epsilon_{r\theta} = M_1 \mu_e [\frac{\partial^2}{\partial r \partial \theta}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_\theta^2 \frac{\partial \psi}{\partial r} - \kappa_r^2 \frac{\partial^2 \chi}{\partial r^2}]$ $\epsilon_{rz} = M_1 \mu_e [2 \frac{\partial^2}{\partial r \partial z}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_z^2 \frac{\partial \psi}{\partial r} - \kappa_r^2 \frac{\partial^2 \chi}{\partial r^2}]$ $\epsilon_{\theta z} = M_1 \mu_e [2 \frac{\partial^2}{\partial \theta \partial z}(\phi + \frac{\partial \psi}{\partial r}) - \kappa_z^2 (\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial^2 \chi}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2})]$ <p>$M_1 = \frac{\omega^2 + i\omega 2\beta}{\omega^2 + 4\beta^2}$</p>	$\epsilon_{rr} = M_1 (\lambda_e \nabla^2 \phi + 2\mu_e [\frac{\partial^2}{\partial r^2}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_r^2 \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})]) + N_1$ $\epsilon_{\theta\theta} = M_1 (\lambda_e \nabla^2 \phi + 2\mu_e [\frac{1}{r} \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}](\phi + \frac{\partial \psi}{\partial r}) - \kappa_\theta^2 \frac{\partial}{\partial r}(\frac{1}{r} \frac{\partial \psi}{\partial r})) + N_1$ $\epsilon_{zz} = M_1 (\lambda_e \nabla^2 \phi + 2\mu_e [\frac{\partial^2}{\partial z^2}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_z^2 \frac{\partial \psi}{\partial r}]) + N_1$ $\epsilon_{r\theta} = M_1 \mu_e [\frac{\partial^2}{\partial r \partial \theta}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_\theta^2 \frac{\partial \psi}{\partial r} - \kappa_r^2 \frac{\partial^2 \chi}{\partial r^2}]$ $\epsilon_{rz} = M_1 \mu_e [2 \frac{\partial^2}{\partial r \partial z}(\phi + \frac{\partial \psi}{\partial r}) + \kappa_z^2 \frac{\partial \psi}{\partial r} - \kappa_r^2 \frac{\partial^2 \chi}{\partial r^2}]$ $\epsilon_{\theta z} = M_1 \mu_e [2 \frac{\partial^2}{\partial \theta \partial z}(\phi + \frac{\partial \psi}{\partial r}) - \kappa_z^2 (\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial^2 \chi}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2})]$ <p>$M_1 = \frac{\omega^2 + i\omega 2\beta}{\omega^2 + 4\beta^2}$</p>
9) HELMHOLTZ'S EQS FOR THE SCALAR & VECTOR POTENTIALS:	9) HELMHOLTZ'S EQS FOR THE SCALAR & VECTOR POTENTIALS:	9) HELMHOLTZ'S EQS FOR THE SCALAR & VECTOR POTENTIALS:
$(\nabla^2 + \kappa_i^2) \phi = 0 \quad , \quad (\nabla^2 + \kappa_i^2) \psi = 0 \quad \& \quad \nabla \cdot \bar{\psi} = 0$	$(\nabla^2 + \kappa_i^2) \phi = 0 \quad , \quad (\nabla^2 + \kappa_i^2) \psi = 0 \quad \& \quad \nabla \cdot \bar{\psi} = 0$	$(\nabla^2 + \kappa_i^2) \phi = 0 \quad , \quad (\nabla^2 + \kappa_i^2) \psi = 0 \quad \& \quad \nabla \cdot \bar{\psi} = 0$

[illegible]

IN SIX CASES OF INTEREST. (SEE REMARK 3) BELOW).

CASE	RELATIONS	REMARKS
LONG CYCLES, HERE THERE IS NO PLANE-STRAIN AMOUNTS TO PLANE-STRAIN PROBLEM.	$\sigma_{kk} = [(1.5\lambda_0 + 2\mu_0) - (\omega(3\lambda_0 + 2\mu_0))] \epsilon_{kk}$ $\sigma_{kk} = \text{BULK STRESS}, \quad \epsilon_{kk} = \text{DILATATION}$	<p>IF AN ISOTROPIC MATERIAL IS CONSIDERED, THEN $\lambda_0 = \frac{2\mu_0}{3}$ AND $\omega = \frac{1}{3}$ AND $\sigma_{kk} = 2\mu_0 \epsilon_{kk}$ AND $\epsilon_{kk} = \frac{1}{2\mu_0} \sigma_{kk}$</p>
$\sigma_{rr} = 0, \quad \sigma_{\theta\theta} = 0, \quad \sigma_{\phi\phi} = 0$	$\begin{cases} \sigma_{rr} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{rr} \\ \sigma_{\theta\theta} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{\theta\theta} \\ \sigma_{\phi\phi} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{\phi\phi} \end{cases}$ $\begin{cases} \Delta = \epsilon_{kk} = \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{\phi\phi} = \nabla \cdot \vec{u} = \Delta_{kk} \text{ OR } \\ \sigma_{rr} = 2(\mu_0 - \omega\mu_0)\epsilon_{rr} \\ \sigma_{\theta\theta} = 2(\mu_0 - \omega\mu_0)\epsilon_{\theta\theta} \\ \sigma_{\phi\phi} = 2(\mu_0 - \omega\mu_0)\epsilon_{\phi\phi} \end{cases}$	<p>IF AN ISOTROPIC MATERIAL IS CONSIDERED, THEN $\lambda_0 = \frac{2\mu_0}{3}$ AND $\omega = \frac{1}{3}$ AND $\sigma_{rr} = 2\mu_0 \epsilon_{rr}$ AND $\epsilon_{rr} = \frac{1}{2\mu_0} \sigma_{rr}$</p>
THESE ARE THE SAME AS IN TABLE 1.	<p>STRESS - STRAIN RELATIONS:</p> $\sigma_{rr} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{rr}$ <p>WHERE THE ϵ_{ij} ARE GIVEN ABOVE IN EQS. 3) IN TERMS OF THE DISPLACEMENT COMPONENTS.</p>	<p>3) STRESS - STRAIN RELATIONS:</p> $\sigma_{rr} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{rr}$
$u_r = 0$	$\begin{cases} u_r = \frac{1}{2} \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r^2 \psi) \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right] \\ u_\theta = \frac{1}{2} \left[\frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial}{\partial r} (r^2 \psi) \right) - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right] \\ u_\phi = \frac{1}{2} \left[\frac{\partial}{\partial \phi} \left(\frac{1}{r} \frac{\partial}{\partial r} (r^2 \psi) \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} \right] \end{cases}$	<p>IF AN ISOTROPIC MATERIAL IS CONSIDERED, THEN $\lambda_0 = \frac{2\mu_0}{3}$ AND $\omega = \frac{1}{3}$ AND $\sigma_{rr} = 2\mu_0 \epsilon_{rr}$ AND $\epsilon_{rr} = \frac{1}{2\mu_0} \sigma_{rr}$</p>
$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2}$	$\sigma_{rr} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{rr}$ <p>WHERE THE ϵ_{ij} ARE GIVEN ABOVE IN EQS. 3) IN TERMS OF THE DISPLACEMENT COMPONENTS.</p>	<p>IF AN ISOTROPIC MATERIAL IS CONSIDERED, THEN $\lambda_0 = \frac{2\mu_0}{3}$ AND $\omega = \frac{1}{3}$ AND $\sigma_{rr} = 2\mu_0 \epsilon_{rr}$ AND $\epsilon_{rr} = \frac{1}{2\mu_0} \sigma_{rr}$</p>
$\sigma_{rr} = 0, \quad \sigma_{\theta\theta} = 0, \quad \sigma_{\phi\phi} = 0$	$\begin{cases} \sigma_{rr} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{rr} \\ \sigma_{\theta\theta} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{\theta\theta} \\ \sigma_{\phi\phi} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{\phi\phi} \end{cases}$	<p>IF AN ISOTROPIC MATERIAL IS CONSIDERED, THEN $\lambda_0 = \frac{2\mu_0}{3}$ AND $\omega = \frac{1}{3}$ AND $\sigma_{rr} = 2\mu_0 \epsilon_{rr}$ AND $\epsilon_{rr} = \frac{1}{2\mu_0} \sigma_{rr}$</p>
$\sigma_{rr} = 0, \quad \sigma_{\theta\theta} = 0, \quad \sigma_{\phi\phi} = 0$	$\begin{cases} \sigma_{rr} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{rr} \\ \sigma_{\theta\theta} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{\theta\theta} \\ \sigma_{\phi\phi} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{\phi\phi} \end{cases}$	<p>IF AN ISOTROPIC MATERIAL IS CONSIDERED, THEN $\lambda_0 = \frac{2\mu_0}{3}$ AND $\omega = \frac{1}{3}$ AND $\sigma_{rr} = 2\mu_0 \epsilon_{rr}$ AND $\epsilon_{rr} = \frac{1}{2\mu_0} \sigma_{rr}$</p>
$\sigma_{rr} = 0, \quad \sigma_{\theta\theta} = 0, \quad \sigma_{\phi\phi} = 0$	$\begin{cases} \sigma_{rr} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{rr} \\ \sigma_{\theta\theta} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{\theta\theta} \\ \sigma_{\phi\phi} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{\phi\phi} \end{cases}$	<p>IF AN ISOTROPIC MATERIAL IS CONSIDERED, THEN $\lambda_0 = \frac{2\mu_0}{3}$ AND $\omega = \frac{1}{3}$ AND $\sigma_{rr} = 2\mu_0 \epsilon_{rr}$ AND $\epsilon_{rr} = \frac{1}{2\mu_0} \sigma_{rr}$</p>
$\sigma_{rr} = 0, \quad \sigma_{\theta\theta} = 0, \quad \sigma_{\phi\phi} = 0$	$\begin{cases} \sigma_{rr} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{rr} \\ \sigma_{\theta\theta} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{\theta\theta} \\ \sigma_{\phi\phi} = (\lambda_0 - \omega\lambda_0)\Delta + 2(\mu_0 - \omega\mu_0)\epsilon_{\phi\phi} \end{cases}$	<p>IF AN ISOTROPIC MATERIAL IS CONSIDERED, THEN $\lambda_0 = \frac{2\mu_0}{3}$ AND $\omega = \frac{1}{3}$ AND $\sigma_{rr} = 2\mu_0 \epsilon_{rr}$ AND $\epsilon_{rr} = \frac{1}{2\mu_0} \sigma_{rr}$</p>

[illegible]

$2\epsilon_{\theta\theta} = \frac{2}{r} \frac{\partial^2}{\partial r \partial z} (\psi + \frac{\partial \psi}{\partial z}) + \frac{\kappa_1^2}{r} \frac{\partial \psi}{\partial r} - \kappa_2^2 \frac{\partial^2 \chi}{\partial r \partial z}$ $2\epsilon_{r\theta} = 2 \frac{\partial^2}{\partial r \partial z} (\psi + \frac{\partial \psi}{\partial z}) + \kappa_1^2 \frac{\partial \psi}{\partial r} + \kappa_2^2 \frac{\partial^2 \chi}{\partial r \partial z}$ $2\epsilon_{r0} = 2 \frac{\partial^2}{\partial r \partial \theta} [\frac{1}{r} (\psi + \frac{\partial \psi}{\partial z})] - \kappa_2^2 (\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2})$	$= \frac{2}{r} \frac{\partial^2 \psi}{\partial r \partial z} + \frac{1}{r} \frac{\partial}{\partial \theta} [2 \frac{\partial^2 \psi}{\partial r^2} - \nabla^2 \psi] + \frac{\partial^2}{\partial r \partial z} (\nabla^2 \chi)$ $= 2 \frac{\partial^2 \psi}{\partial r \partial z} + \frac{2}{r} [2 \frac{\partial^2 \psi}{\partial r^2} - \nabla^2 \psi] - \frac{1}{r} \frac{\partial^2}{\partial \theta^2} (\nabla^2 \chi)$ $= 2 \frac{\partial}{\partial r} [\frac{1}{r} \frac{\partial \psi}{\partial \theta}] + \frac{2}{r} \frac{\partial}{\partial \theta} [\frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{\partial \psi}{\partial z}] + [\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}] (\nabla^2 \chi)$	$2\epsilon_{\theta\theta} =$ $\epsilon_{\theta\theta} =$
<p>7) FIELD EQUATIONS:</p> <p>i): $\nabla^2 \vec{u} + [1 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}] \nabla(\vec{\sigma} \cdot \vec{u}) + \kappa_2^2 \vec{u} = 0$ WHERE,</p> <p>$\kappa_2^2 = \rho\omega^2/(\mu_e - i\omega\mu_v)$. THIS EQ. CAN BE OPENED UP BY MEANS OF</p> <p>$\nabla^2 \vec{u} = [\nabla^2 u_r - \frac{u_r}{r} - \frac{2}{r} \frac{\partial u_\theta}{\partial \theta}] \hat{e}_r + [\nabla^2 u_\theta - \frac{u_\theta}{r} + \frac{2}{r} \frac{\partial u_r}{\partial \theta}] \hat{e}_\theta + [\nabla^2 u_z] \hat{e}_z$</p> <p>AND $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$</p> <p>THE GRADIENT OPERATOR & THE DILATATION Δ ARE GIVEN ABOVE.</p> <p>ii): $-\vec{\sigma} \cdot \nabla(\vec{\sigma} \cdot \vec{u}) + [2 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}] \vec{\sigma}(\vec{\sigma} \cdot \vec{u}) + \kappa_2^2 \vec{u} = 0$</p> <p>NOTE: $\nabla^2 \vec{u} = \vec{\sigma}(\vec{\sigma} \cdot \vec{u}) - \vec{\sigma} \cdot \nabla(\vec{\sigma} \cdot \vec{u})$</p>	<p>7) FIELD EQUATIONS:</p> <p>i): $(1 - \frac{i\omega}{2\beta}) (\nabla^2 \vec{u} + \frac{\lambda_e + \mu_e}{\mu_e} \vec{\sigma}(\vec{\sigma} \cdot \vec{u}) + \frac{\omega^2}{c^2} \vec{u}) + \frac{\partial}{\partial z} [\nabla^2 \vec{u} + \frac{\partial}{\partial z} (\vec{\sigma} \cdot \vec{u}) + \frac{\omega^2}{c^2} \vec{u}] + \frac{\omega^2}{c^2} \vec{u} + i\omega(\frac{3\alpha + 2\beta}{c^2}) \vec{u} = 0$</p> <p>OR IN COMPONENT FORM,</p> <p>$[\nabla^2 u_r - \frac{u_r}{r} - \frac{2}{r} \frac{\partial u_\theta}{\partial \theta}] + \frac{\partial}{\partial z} \frac{\partial u_z}{\partial r} + W u_r = 0,$</p> <p>$[\nabla^2 u_\theta - \frac{u_\theta}{r} + \frac{2}{r} \frac{\partial u_r}{\partial \theta}] + \frac{\partial}{\partial z} \frac{\partial u_z}{\partial \theta} + W u_\theta = 0, \quad [\nabla^2 u_z] + \frac{\partial}{\partial z} \frac{\partial u_z}{\partial z} + W u_z = 0,$</p> <p>WHERE, $V = \frac{\alpha\mu_e + (\lambda_e + \mu_e)(2\beta - i\omega)}{\mu_e(3\alpha + 2\beta - i\omega)}, W = \frac{\omega^2(3\alpha + 4\beta - i\omega) + 2\beta\omega(3\alpha + 2\beta)}{c^2(3\alpha + 2\beta - i\omega)}$</p> <p>ii): $(1 - \frac{i\omega}{2\beta}) [\frac{\lambda_e + \mu_e}{\mu_e} \vec{\sigma}(\vec{\sigma} \cdot \vec{u}) - \vec{\sigma} \cdot \nabla(\vec{\sigma} \cdot \vec{u}) + \frac{\omega^2}{c^2} \vec{u}] + \frac{\partial}{\partial z} [\frac{\partial}{\partial z} (\vec{\sigma} \cdot \vec{u}) - \vec{\sigma} \cdot \nabla(\vec{\sigma} \cdot \vec{u}) + \frac{\omega^2}{c^2} \vec{u}] + \frac{\omega^2}{c^2} [1 + i(\frac{3\alpha + 2\beta}{c^2})] \vec{u} = 0$</p> <p>$\Delta = \vec{\sigma} \cdot \vec{u}$ IS GIVEN IN 5).</p>	<p>7) FIELD EQUATIONS:</p> <p>i): $[\mu_e - i\omega\mu_v]$</p> <p>$[\mu_e - i\omega\mu_v]$</p> <p>$\Delta = \frac{\partial}{\partial z}$</p> <p>NOTE:</p> <p>FORM</p>
<p>8) STRESS - INDEPENDENT POTENTIAL RELATIONS:</p> <p>$\tau_{rr} = (\lambda_e - i\omega\lambda_v) \nabla^2 \psi + 2(\mu_e - i\omega\mu_v) [\frac{\partial^2}{\partial r^2} (\psi + \frac{\partial \psi}{\partial z}) + \kappa_2^2 \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial \psi}{\partial \theta})]$</p> <p>$\tau_{\theta\theta} = (\lambda_e - i\omega\lambda_v) \nabla^2 \psi + 2(\mu_e - i\omega\mu_v) [\frac{1}{r} \frac{\partial^2}{\partial \theta^2} (\psi + \frac{\partial \psi}{\partial z}) - \kappa_2^2 \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial \psi}{\partial \theta})]$</p> <p>$\tau_{zz} = (\lambda_e - i\omega\lambda_v) \nabla^2 \psi + 2(\mu_e - i\omega\mu_v) [\frac{\partial^2}{\partial z^2} (\psi + \frac{\partial \psi}{\partial z}) + \kappa_2^2 \frac{\partial \psi}{\partial z}]$</p> <p>$\tau_{r\theta} = (\mu_e - i\omega\mu_v) [\frac{2}{r} \frac{\partial^2}{\partial r \partial \theta} (\psi + \frac{\partial \psi}{\partial z}) + \frac{\kappa_2^2}{r} \frac{\partial \psi}{\partial r} - \kappa_2^2 \frac{\partial^2 \chi}{\partial r \partial z}]$</p> <p>$\tau_{\theta z} = (\mu_e - i\omega\mu_v) [2 \frac{\partial^2}{\partial r \partial z} (\psi + \frac{\partial \psi}{\partial z}) + \kappa_2^2 \frac{\partial \psi}{\partial r} + \frac{\kappa_2^2}{r} \frac{\partial^2 \chi}{\partial \theta^2}]$</p> <p>$\tau_{r0} = (\mu_e - i\omega\mu_v) [2 \frac{\partial^2}{\partial r \partial \theta} (\frac{1}{r} (\psi + \frac{\partial \psi}{\partial z})) - \kappa_2^2 \{\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2}\}]$</p> <p>THESE ARE EQS 5) WITH $\Delta = \nabla^2 \psi + \kappa_1^2 \psi$ AND ϵ_θ AS GIVEN IN 6).</p>	<p>8) STRESS - INDEPENDENT POTENTIAL RELATIONS:</p> <p>$\tau_{rr} = M_1 \{ \lambda_e \nabla^2 \psi + 2\mu_e [\frac{\partial^2}{\partial r^2} (\psi + \frac{\partial \psi}{\partial z}) + \kappa_2^2 \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial \psi}{\partial \theta})] \} + N_1$</p> <p>$\tau_{\theta\theta} = M_1 \{ \lambda_e \nabla^2 \psi + 2\mu_e [\frac{1}{r} \frac{\partial^2}{\partial \theta^2} (\psi + \frac{\partial \psi}{\partial z}) - \kappa_2^2 \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial \psi}{\partial \theta})] \} + N_1$</p> <p>$\tau_{zz} = M_1 \{ \lambda_e \nabla^2 \psi + 2\mu_e [\frac{\partial^2}{\partial z^2} (\psi + \frac{\partial \psi}{\partial z}) + \kappa_2^2 \frac{\partial \psi}{\partial z}] \} + N_1$</p> <p>$\tau_{r\theta} = M_1 \mu_e [\frac{2}{r} \frac{\partial^2}{\partial r \partial \theta} (\psi + \frac{\partial \psi}{\partial z}) + \frac{\kappa_2^2}{r} \frac{\partial \psi}{\partial r} - \kappa_2^2 \frac{\partial^2 \chi}{\partial r \partial z}]$</p> <p>$\tau_{\theta z} = M_1 \mu_e [2 \frac{\partial^2}{\partial r \partial z} (\psi + \frac{\partial \psi}{\partial z}) + \kappa_2^2 \frac{\partial \psi}{\partial r} + \frac{\kappa_2^2}{r} \frac{\partial^2 \chi}{\partial \theta^2}]$</p> <p>$\tau_{r0} = M_1 \mu_e [2 \frac{\partial^2}{\partial r \partial \theta} (\frac{1}{r} (\psi + \frac{\partial \psi}{\partial z})) - \kappa_2^2 \{\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2}\}]$</p> <p>$\tau_{\theta z} = M_1 \mu_e [2 \frac{\partial^2}{\partial r \partial z} (\psi + \frac{\partial \psi}{\partial z}) + \kappa_2^2 \frac{\partial \psi}{\partial r} + \frac{\kappa_2^2}{r} \frac{\partial^2 \chi}{\partial \theta^2}]$, $M_1 = \frac{\omega^2 \mu_e 2\beta}{\omega^2 + 4\beta^2}$</p> <p>$N_1 = \frac{\alpha^2 \mu_e^2 \beta (\frac{3}{2} \lambda_e - 2\mu_e) \{ \frac{\omega^2(3\alpha + 4\beta - i\omega) + 2\beta\omega(3\alpha + 2\beta - i\omega)}{\omega^2 + 4\beta^2} \}}{\omega^2 + 4\beta^2}$</p>	<p>8) STRESS</p> <p>$\tau_{rr} = -[$</p> <p>$\tau_{\theta\theta} = -[$</p> <p>$\tau_{zz} = -[$</p> <p>$\tau_{r\theta} = -[$</p> <p>$\tau_{\theta z} = -[$</p> <p>$\tau_{r0} = -[$</p> <p>TO HAVE</p>
<p>9) HELMHOLTZ'S EQS. FOR THE SCALAR & VECTOR POTENTIALS:</p> <p>i): $(\nabla^2 + \kappa_1^2) \psi = 0$, ii): $(\nabla^2 + \kappa_2^2) \vec{\chi} = 0$ & $\vec{\sigma} \cdot \vec{\chi} = 0$</p> <p>WHERE, $\kappa_1^2 = \omega^2/c_e^2 (1 - i\omega M)$, $\kappa_2^2 = \omega^2/c_s^2 (1 - i\omega N)$, $M = \frac{\lambda_e + 2\mu_e}{\lambda_e + 2\mu_e}$, $N = \frac{\mu_e}{\mu_e}$, $c_e^2 = \frac{\lambda_e + 2\mu_e}{\rho}$, $c_s^2 = \frac{\mu_e}{\rho}$</p> <p>NOTE THESE ARE HELMHOLTZ'S EQS. WITH COMPLEX κ'S.</p>	<p>9) HELMHOLTZ'S EQS. FOR THE SCALAR & VECTOR POTENTIALS:</p> <p>i): $(\nabla^2 + \kappa_1^2) \psi = 0$, ii): $(\nabla^2 + \kappa_2^2) \vec{\chi} = 0$ & $\vec{\sigma} \cdot \vec{\chi} = 0$</p> <p>WHERE, $\kappa_1^2 = \frac{\omega^2}{c_e^2} \{ \frac{\lambda_e + 4\beta + 4\alpha(3\alpha + 4\beta - i\omega)}{\omega^2 + 4\beta^2} \}$</p> <p>$+ i\omega \{ \frac{\omega^2(3\alpha + 4\beta - i\omega) + 2\beta\omega(3\alpha + 2\beta - i\omega)}{\omega^2 + 4\beta^2} \}$</p> <p>AND, $\kappa_2^2 = \frac{\omega^2}{c_s^2} [1 + i(\frac{3\alpha + 2\beta}{c_s^2})]$, $c_e^2 = \lambda_e/\rho$, $c_s^2 = (\lambda_e + 2\mu_e)/\rho$</p>	<p>9) HELMHOLTZ'S EQS.</p> <p>i): $(\nabla^2 + \kappa_1^2) \psi = 0$</p> <p>WHERE</p> <p>$M = \frac{\lambda_e + 2\mu_e}{\lambda_e + 2\mu_e}$</p> <p>WHERE</p> <p>IS EQUI</p>
<p>10) HELMHOLTZ'S EQS. FOR THE SCALAR & VECTOR POTENTIALS:</p> <p>i): $(\nabla^2 + \kappa_1^2) \psi = 0$, ii): $(\nabla^2 + \kappa_2^2) \vec{\chi} = 0$, iii): $(\nabla^2 + \kappa_3^2) \vec{\chi} = 0$</p> <p>WHERE $\kappa_1^2 = \omega^2/c_e^2 (1 - i\omega M)$, $\kappa_2^2 = \omega^2/c_s^2 (1 - i\omega N)$</p> <p>AND M, N, c_e^2, c_s^2 ARE AS GIVEN ABOVE IN 9).</p>	<p>10) HELMHOLTZ'S EQS. FOR THE SCALAR & VECTOR POTENTIALS:</p> <p>i): $(\nabla^2 + \kappa_1^2) \psi = 0$, ii): $(\nabla^2 + \kappa_2^2) \vec{\chi} = 0$, iii): $(\nabla^2 + \kappa_3^2) \vec{\chi} = 0$</p> <p>WHERE κ_1^2 AND κ_2^2 ARE AS GIVEN ABOVE IN 9).</p> <p>THE LIMIT $c_e^2 \rightarrow c_s^2$ (OR $\mu_e \rightarrow \lambda_e$) IMPLIES THAT THE MATERIAL IS RUBBER. ONLY THEN WILL κ_1^2 DEPEND ON c_s^2 ALONE.</p>	<p>10) HELMHOLTZ'S EQS.</p> <p>i): $(\nabla^2 + \kappa_1^2) \psi = 0$</p> <p>WHERE</p> <p>IS IS</p>
<p>11) REMARKS:</p> <p>IF IT IS KNOWN THAT THE SCALAR (OR VECTOR) POTENTIAL ψ (OR $\vec{\chi}$) IS THE SOLUTION OF $(\nabla^2 + \kappa_1^2) \psi = 0$ (OR $(\nabla^2 + \kappa_2^2) \vec{\chi} = 0$), WHICH IS THE SOLUTION OF $\vec{\sigma} \cdot \nabla(\vec{\sigma} \cdot \vec{u}) - \kappa_2^2 \vec{u} = 0$, IS GIVEN BY,</p> <p>$\vec{u} = \vec{\sigma}(\psi \hat{e}_r) + \vec{\sigma} \cdot \nabla(\vec{\chi} \hat{e}_z) = (\vec{\sigma} \psi) \cdot \hat{e}_r + \vec{\sigma}(\frac{\partial \chi}{\partial z}) = \hat{e}_r \kappa_2^2 \chi$ (A) IN CYLINDRICAL COORDS ONLY.</p> <p>HERE $\psi(r, \theta, z)$ AND $\chi(r, \theta, z)$ ARE TWO SCALAR FUNCTIONS WHICH SATISFY SCALAR HELMHOLTZ'S EQS. (10) (i) & (ii).</p> <p>EQ. (A) IN COMPONENT FORM, IS:</p> <p>$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \frac{\partial \chi}{\partial z}$, $u_\theta = -\frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial \chi}{\partial \theta}$, $u_z = \frac{\partial \psi}{\partial z} + \kappa_2^2 \chi$ (B).</p> <p>WHICH RELATE THE THREE CYLINDRICAL COMPONENTS OF THE VECTOR POTENTIAL \vec{u} TO THE TWO SCALAR FUNCTIONS ψ AND χ.</p> <p>SUBSTITUTING \vec{u} FROM EQ. (A) INTO $\vec{u} = \vec{\sigma} \cdot \nabla \psi$ YIELDS THE DISPLACEMENT-INDEPENDENT POTENTIAL RELATIONS GIVEN ABOVE IN 4). NOTE THAT \vec{u} IS NOT $\vec{\sigma} \cdot \nabla \psi$. IN THE ABSENCE OF VISCOSITY (IE, $M=N=0$, IE, ELASTICITY) EQ. (A) ABOVE IS THE SCALAR SOLUTION OF THE VECTOR HELMHOLTZ EQ. NOW WITH A REAL, RATHER THAN COMPLEX PROPAGATION CONSTANT. HENCE, FOR VIBRATIONS HARMONIC IN TIME, DAMPING IS ACCOUNTED FOR BY MEANS OF COMPLEX PROPAGATION CONSTANTS κ_1^2 & κ_2^2 RELATED TO THE ELASTIC AND VISCOUS MATERIAL CONSTANTS BY THE FORMULAS ABOVE IN 9).</p> <p>κ_1^2 IS ASSOCIATED WITH LONGITUDINAL OR DILATATIONAL WAVES AND κ_2^2 WITH TRANSVERSE OR SHEAR WAVES.</p> <p>IN CYLINDRICAL COORDS. ONE CAN CHECK THAT, $\vec{\sigma} \cdot \nabla(\vec{\sigma} \cdot \vec{u}) = \vec{\sigma} \cdot \nabla(\vec{\sigma} \cdot \vec{u})$ AND THAT, $(\vec{\sigma} \cdot \nabla \psi) = \vec{\sigma} \cdot \nabla(\vec{\sigma} \cdot \vec{u})$.</p>	<p>11) REMARKS:</p> <p>IF IT IS KNOWN THAT THE SCALAR (OR VECTOR) POTENTIAL ψ (OR $\vec{\chi}$) IS THE SOLUTION OF $(\nabla^2 + \kappa_1^2) \psi = 0$ (OR $(\nabla^2 + \kappa_2^2) \vec{\chi} = 0$), WHICH IS THE SOLUTION OF $\vec{\sigma} \cdot \nabla(\vec{\sigma} \cdot \vec{u}) - \kappa_2^2 \vec{u} = 0$, IS GIVEN BY,</p> <p>$\vec{u} = \vec{\sigma}(\psi \hat{e}_r) + \vec{\sigma} \cdot \nabla(\vec{\chi} \hat{e}_z) = (\vec{\sigma} \psi) \cdot \hat{e}_r + \vec{\sigma}(\frac{\partial \chi}{\partial z}) = \hat{e}_r \kappa_2^2 \chi$ (A) IN CYLINDRICAL COORDS ONLY.</p> <p>HERE $\psi(r, \theta, z)$ AND $\chi(r, \theta, z)$ ARE TWO SCALAR FUNCTIONS WHICH SATISFY SCALAR HELMHOLTZ'S EQS. (10) (i) & (ii).</p> <p>EQ. (A) IN COMPONENT FORM, IS:</p> <p>$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \frac{\partial \chi}{\partial z}$, $u_\theta = -\frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial \chi}{\partial \theta}$, $u_z = \frac{\partial \psi}{\partial z} + \kappa_2^2 \chi$ (B).</p> <p>WHICH RELATE THE THREE CYLINDRICAL COMPONENTS OF THE VECTOR POTENTIAL \vec{u} TO THE TWO SCALAR FUNCTIONS ψ AND χ.</p> <p>SUBSTITUTING \vec{u} FROM EQ. (A) INTO $\vec{u} = \vec{\sigma} \cdot \nabla \psi$ YIELDS THE DISPLACEMENT-INDEPENDENT POTENTIAL RELATIONS GIVEN ABOVE IN 4). NOTE THAT \vec{u} IS NOT $\vec{\sigma} \cdot \nabla \psi$. IN THE ABSENCE OF VISCOSITY (IE, $M=N=0$, IE, ELASTICITY) EQ. (A) ABOVE IS THE SCALAR SOLUTION OF THE VECTOR HELMHOLTZ EQ. NOW WITH A REAL, RATHER THAN COMPLEX PROPAGATION CONSTANT. 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(A) IN COMPONENT FORM, IS:</p> <p>$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \frac{\partial \chi}{\partial z}$, $u_\theta = -\frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial \chi}{\partial \theta}$, $u_z = \frac{\partial \psi}{\partial z} + \kappa_2^2 \chi$ (B).</p> <p>WHICH RELATE THE THREE CYLINDRICAL COMPONENTS OF THE VECTOR POTENTIAL \vec{u} TO THE TWO SCALAR FUNCTIONS ψ AND χ.</p> <p>SUBSTITUTING \vec{u} FROM EQ. (A) INTO $\vec{u} = \vec{\sigma} \cdot \nabla \psi$ YIELDS THE DISPLACEMENT-INDEPENDENT POTENTIAL RELATIONS GIVEN ABOVE IN 4). NOTE THAT \vec{u} IS NOT $\vec{\sigma} \cdot \nabla \psi$. IN THE ABSENCE OF VISCOSITY (IE, $M=N=0$, IE, ELASTICITY) EQ. (A) ABOVE IS THE SCALAR SOLUTION OF THE VECTOR HELMHOLTZ EQ. NOW WITH A REAL, RATHER THAN COMPLEX PROPAGATION CONSTANT. HENCE, FOR VIBRATIONS HARMONIC IN TIME, DAMPING IS ACCOUNTED FOR BY MEANS OF COMPLEX PROPAGATION CONSTANTS κ_1^2 & κ_2^2 RELATED TO THE ELASTIC AND VISCOUS MATERIAL CONSTANTS BY THE FORMULAS ABOVE IN 9).</p> <p>κ_1^2 IS ASSOCIATED WITH LONGITUDINAL OR DILATATIONAL WAVES AND κ_2^2 WITH TRANSVERSE OR SHEAR WAVES.</p> <p>IN CYLINDRICAL COORDS. ONE CAN CHECK THAT, $\vec{\sigma} \cdot \nabla(\vec{\sigma} \cdot \vec{u}) = \vec{\sigma} \cdot \nabla(\vec{\sigma} \cdot \vec{u})$ AND THAT, $(\vec{\sigma} \cdot \nabla \psi) = \vec{\sigma} \cdot \nabla(\vec{\sigma} \cdot \vec{u})$.</p>
<p>12) REMARKS:</p> <p>a) THE BASIC RESULT OF THIS TABLE, AND THAT IN WHY IT WAS ORGANIZED AS SHOWN ABOVE, IS THAT IF WE INTRODUCE INDEPENDENT SCALAR POTENTIALS SATISFIED IN EACH ONE IN THE CASES CONSIDERED, AND THEN THE DISPLACEMENT AND STRESS FIELD COMPONENTS ARE GIVEN IN TERMS OF THE PROBLEMS, AND WHICH IS COMMON IN OTHER DISCIPLINES (IE, ELECTRODYNAMICS), IS APPARENTLY NOVEL IN VISCOELASTICITY, AND WITH ELECTRODYNAMIC THEORY WILL QUICKLY NOTICE THE ANALOGY OF THE PRESENTATION IN THIS TABLE WITH THAT OF ELECTROMAGNETISM.</p> <p>b) IT IS EVIDENT THAT IN ALL THE CASES SHOWN ABOVE, WE RECOVER THE COMPLEX RESULTS OF DYNAMICAL ELASTICITY, IN THE ABSENCE OF VISCOSITY (IE, $M=N=0$).</p> <p>c) IN CYLINDRICAL COORDS, THE PLANE-STRAIN RESULTS CAN BE QUICKLY OBTAINED FROM THE PLANE-STRAIN RESULTS (CASES 2a & 2b) ABOVE BY SETTING $\frac{\partial}{\partial z} = 0$.</p> <p>d) IN THE CASE OF PLANE-STRAIN (ON THE OTHER HAND, OF c) ABOVE) WITH AXIAL SYMMETRY ABOUT THE Z-AXIS, CAN BE OBTAINED FROM 2b) BY SETTING $\frac{\partial}{\partial \theta} = 0$.</p> <p>e) A CASE OF SOME INTEREST WHICH REDUCES THE TWO SCALAR CONSTANTS TO ONE, OCCURS WHEN THE BULK VISCOSITY IS ZERO OR NEARLY SO. THIS IS AN</p>	<p>12) REMARKS</p> <p>a) THE BASIC RESULT OF THIS TABLE, AND THAT IN WHY IT WAS ORGANIZED AS SHOWN ABOVE, IS THAT IF WE INTRODUCE INDEPENDENT SCALAR POTENTIALS SATISFIED IN EACH ONE IN THE CASES CONSIDERED, AND THEN THE DISPLACEMENT AND STRESS FIELD COMPONENTS ARE GIVEN IN TERMS OF THE PROBLEMS, AND WHICH IS COMMON IN OTHER DISCIPLINES (IE, ELECTRODYNAMICS), IS APPARENTLY NOVEL IN VISCOELASTICITY, AND WITH ELECTRODYNAMIC THEORY WILL QUICKLY NOTICE THE ANALOGY OF THE PRESENTATION IN THIS TABLE WITH THAT OF ELECTROMAGNETISM.</p> <p>b) IT IS EVIDENT THAT IN ALL THE CASES SHOWN ABOVE, WE RECOVER THE COMPLEX RESULTS OF DYNAMICAL ELASTICITY, IN THE ABSENCE OF VISCOSITY (IE, $M=N=0$).</p> <p>c) IN CYLINDRICAL COORDS, THE PLANE-STRAIN RESULTS CAN BE QUICKLY OBTAINED FROM THE PLANE-STRAIN RESULTS (CASES 2a & 2b) ABOVE BY SETTING $\frac{\partial}{\partial z} = 0$.</p> <p>d) IN THE CASE OF PLANE-STRAIN (ON THE OTHER HAND, OF c) ABOVE) WITH AXIAL SYMMETRY ABOUT THE Z-AXIS, CAN BE OBTAINED FROM 2b) BY SETTING $\frac{\partial}{\partial \theta} = 0$.</p> <p>e) A CASE OF SOME INTEREST WHICH REDUCES THE TWO SCALAR CONSTANTS TO ONE, OCCURS WHEN THE BULK VISCOSITY IS ZERO OR NEARLY SO. THIS IS AN</p>	<p>12) REMARKS</p> <p>a) THE BASIC RESULT OF THIS TABLE, AND THAT IN WHY IT WAS ORGANIZED AS SHOWN ABOVE, IS THAT IF WE INTRODUCE INDEPENDENT SCALAR POTENTIALS SATISFIED IN EACH ONE IN THE CASES CONSIDERED, AND THEN THE DISPLACEMENT AND STRESS FIELD COMPONENTS ARE GIVEN IN TERMS OF THE PROBLEMS, AND WHICH IS COMMON IN OTHER DISCIPLINES (IE, ELECTRODYNAMICS), IS APPARENTLY NOVEL IN VISCOELASTICITY, AND WITH ELECTRODYNAMIC THEORY WILL QUICKLY NOTICE THE ANALOGY OF THE PRESENTATION IN THIS TABLE WITH THAT OF ELECTROMAGNETISM.</p> <p>b) IT IS EVIDENT THAT IN ALL THE CASES SHOWN ABOVE, WE RECOVER THE COMPLEX RESULTS OF DYNAMICAL ELASTICITY, IN THE ABSENCE OF VISCOSITY (IE, $M=N=0$).</p> <p>c) IN CYLINDRICAL COORDS, THE PLANE-STRAIN RESULTS CAN BE QUICKLY OBTAINED FROM THE PLANE-STRAIN RESULTS (CASES 2a & 2b) ABOVE BY SETTING $\frac{\partial}{\partial z} = 0$.</p> <p>d) IN THE CASE OF PLANE-STRAIN (ON THE OTHER HAND, OF c) ABOVE) WITH AXIAL SYMMETRY ABOUT THE Z-AXIS, CAN BE OBTAINED FROM 2b) BY SETTING $\frac{\partial}{\partial \theta} = 0$.</p> <p>e) A CASE OF SOME INTEREST WHICH REDUCES THE TWO SCALAR CONSTANTS TO ONE, OCCURS WHEN THE BULK VISCOSITY IS ZERO OR NEARLY SO. THIS IS AN</p>

$$2\epsilon_{r0} = 2 \frac{\partial}{\partial r} \left(\frac{\phi}{r} \right) - \kappa_1^2 \left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) \quad (\text{BY } \psi_r, \text{ i.e., } -)$$

$$\epsilon_{r\theta} = \epsilon_{\theta r} = \epsilon_{\theta\theta} = 0$$

$$2\epsilon_{r0} = 2 \frac{\partial}{\partial r} \left(\frac{\phi}{r} \right) - \left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right)$$

$$\epsilon_{r\theta} = \epsilon_{\theta r} = \epsilon_{\theta\theta} = 0$$

FIELD EQS. AT $r=a$:

$$\begin{cases} \sigma_r = \lambda_e + 2\mu_e \left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) \\ \sigma_\theta = \lambda_e + 2\mu_e \left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) \end{cases}$$

$$\Delta = \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$$

NOTE: $\kappa_1^2 = \frac{\rho \omega^2}{\mu_e} (1 + i \frac{\eta_e}{\omega \mu_e})$
FORM (1) IS NOT AS USEFUL.

FIELD EQS. AT $r=a$:

$$\begin{cases} \sigma_r = \lambda_e + 2\mu_e \left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) \\ \sigma_\theta = \lambda_e + 2\mu_e \left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) \end{cases}$$

$$\Delta = \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}, \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, \quad \kappa_2^2 = \frac{\rho \omega^2}{\mu_e - i \eta_e \omega}$$

FORM (1) IS NOT AS USEFUL.

9) DISPLACEMENT-DEPENDENT POTENTIAL RELATIONS:

$$\begin{aligned} \epsilon_{rr} &= -(\lambda_e - i\omega \lambda_e) \kappa_1^2 \phi + 2(\mu_e - i\omega \mu_e) \left[\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] \\ \epsilon_{\theta\theta} &= -(\lambda_e - i\omega \lambda_e) \kappa_1^2 \phi + 2(\mu_e - i\omega \mu_e) \left[\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] \\ \epsilon_{r\theta} &= -(\lambda_e - i\omega \lambda_e) \kappa_1^2 \phi \\ \epsilon_{r0} &= \epsilon_{\theta 0} = 0 \\ \epsilon_{r\theta} &= (\mu_e - i\omega \mu_e) \left[2 \frac{\partial^2 \phi}{\partial r^2} - \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] \end{aligned}$$

TO HAVE THESE IN TERMS OF ϕ & ψ , WE REPLACE $\kappa_1^2 \chi$ BY ψ_r .

9) DISPLACEMENT-DEPENDENT POTENTIAL RELATIONS:

$$\begin{aligned} \epsilon_{rr} &= -(\lambda_e - i\omega \lambda_e) \kappa_1^2 \phi + 2\mu_e \left[\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] + N_1 \\ \epsilon_{\theta\theta} &= M_1 \{ -\lambda_e \kappa_1^2 \phi + 2\mu_e \left[\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] \} + N_1 \\ \epsilon_{r\theta} &= M_1 \{ -\lambda_e \kappa_1^2 \phi \} + N_1, \quad \epsilon_{r0} = 0, \quad \epsilon_{\theta 0} = 0 \\ \epsilon_{r\theta} &= \mu_e M_1 \left[2 \frac{\partial^2 \phi}{\partial r^2} - \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] \end{aligned}$$

WHERE,

$$N_1 = \frac{\lambda_e \mu_e}{\omega^2 + 4\mu_e^2} \left[\frac{\omega^2 (3\alpha + 4\beta) - (i\omega 2\beta (3\alpha + 2\beta - \omega^2/2\beta))}{\omega^2 + 4\mu_e^2} \right]$$

$$M_1 = \frac{\omega^2 - i\omega 2\beta}{\omega^2 + 4\mu_e^2} \quad \left\{ \begin{array}{l} \text{TO HAVE THESE IN TERMS OF } \phi \text{ AND } \psi_r \\ \text{WE REPLACE } \kappa_1^2 \chi \text{ BY } \psi_r \end{array} \right.$$

9) HELMHOLTZ'S EQUATIONS FOR SCALAR AND VECTOR POTENTIALS:

$$\begin{aligned} (i) (\nabla^2 + \kappa_1^2) \phi &= 0, \quad (ii) (\nabla^2 + \kappa_2^2) \vec{\psi} = 0 \quad \& \quad \vec{\nabla} \cdot \vec{\psi} = 0 \\ \text{WHERE: } \kappa_1^2 &= \frac{\rho \omega^2}{\mu_e} (1 + i \frac{\eta_e}{\omega \mu_e}), \quad \kappa_2^2 = \frac{\rho \omega^2}{\mu_e - i \eta_e \omega} \\ M &= \frac{\lambda_e + 2\mu_e}{\lambda_e + \mu_e}, \quad N = \frac{\mu_e}{\lambda_e + \mu_e}, \quad C_1^2 = \frac{\lambda_e + 2\mu_e}{\lambda_e + \mu_e}, \quad C_2^2 = \frac{\mu_e}{\lambda_e + \mu_e} \\ \text{HERE } \vec{\psi} &= \vec{e}_r \psi_r(r, \theta) = -\vec{e}_\theta \nabla^2 \chi(r, \theta) = \vec{e}_\theta \kappa_1^2 \chi, \text{ HENCE (1) ABOVE} \\ &\text{IS EQUIVALENT TO EITHER ALTERNATIVE FORM GIVEN IN (10).} \end{aligned}$$

9) HELMHOLTZ'S EQS. FOR SCALAR & VECTOR POTENTIALS ϕ & $\vec{\psi}$:

$$\begin{aligned} (i) (\nabla^2 + \kappa_1^2) \phi &= 0, \quad (ii) (\nabla^2 + \kappa_2^2) \vec{\psi} = 0 \quad \& \quad \vec{\nabla} \cdot \vec{\psi} = 0 \\ \kappa_1^2 &= \frac{\rho \omega^2}{\mu_e} \left\{ \frac{\omega^2 - \frac{1}{2} \frac{\omega^2}{\omega^2 + 4\mu_e^2} + 4\mu_e (3\alpha + 4\beta) \frac{C_1^2/C_2^2}{\omega^2 + 4\mu_e^2}}{\omega^2 + 4\mu_e^2} \right\} + \\ &+ \omega \left\{ \frac{(\omega^2 + 4\mu_e^2) \frac{C_1^2/C_2^2}{\omega^2 + 4\mu_e^2} + 4\mu_e (2\beta (3\alpha + 2\beta) - \omega^2/2\beta) \frac{C_1^2/C_2^2}{\omega^2 + 4\mu_e^2}}{\omega^2 + 4\mu_e^2} \right\} \frac{\omega^2}{C_1^2/C_2^2} \left[1 + i \frac{(\eta_e/\omega \mu_e)}{C_1^2/C_2^2} \right] \\ \kappa_2^2 &= \frac{\rho \omega^2}{\mu_e - i \eta_e \omega} \left[1 + i \frac{2\beta}{\omega} \right] \end{aligned}$$

10) HELMHOLTZ'S EQUATION FOR INDEPENDENT SCALAR POTENTIALS:

$$\begin{aligned} (i) (\nabla^2 + \kappa_1^2) \phi &= 0, \quad (ii) (\nabla^2 + \kappa_2^2) \psi_r = 0 \\ \text{WHERE } \kappa_1^2, \kappa_2^2, M, N, C_1^2, C_2^2, &\text{ ARE AS GIVEN ABOVE IN (9).} \\ (1) \text{ IS EQUIVALENT TO: } &-\nabla^2 [(\nabla^2 + \kappa_2^2) \chi] = 0 \end{aligned}$$

10) HELMHOLTZ'S EQS. FOR INDEPENDENT SCALAR POTENTIALS:

$$\begin{aligned} (i) (\nabla^2 + \kappa_1^2) \phi &= 0, \quad (ii) (\nabla^2 + \kappa_2^2) \psi_r = 0 \\ \text{WHERE } \kappa_1^2 \& \kappa_2^2 \text{ ARE AS GIVEN ABOVE IN (9). SINCE } \psi_r &= \kappa_1^2 \chi \\ \text{EQ. (1) IS EQUIVALENT TO } &-\nabla^2 [(\nabla^2 + \kappa_2^2) \chi] = 0. \\ \text{THE CONDITION } C_1^2 \ll C_2^2 &\text{ MEANS THE MATERIAL IS RUBBER.} \\ \text{ONLY FOR } C_1^2 \ll C_2^2 &\text{ WILL } \kappa_1^2 \text{ DEPEND ON } C_1^2 \text{ ALONE.} \end{aligned}$$

11) SOLENOIDAL SOLUTION OF VECTOR HELMHOLTZ'S EQUATION IN PLANE POLAR COORDINATES:

THE SOLUTION FOLLOWS THAT GIVEN TO THE LEFT FOR GENERAL CYLINDRICAL COORDINATES BUT NOW WITHOUT THE z -DEPENDENCE. HENCE, ONE CAN ALSO SHOW THAT THE SOLENOIDAL SOLUTION OF $(\nabla^2 + \kappa_2^2) \vec{\psi} = 0$, WHICH IS ALSO THE SOLUTION OF $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\psi}) = -\kappa_2^2 \vec{\psi}$, IS GIVEN BY,

$$\vec{\psi} = \vec{\nabla}_\theta \times (\vec{\nabla}_\theta \times (\vec{e}_\theta \chi)) = \vec{e}_\theta \kappa_2^2 \chi \quad (A)$$

WHERE $\chi(r, \theta)$ IS A SCALAR FUNCTION SATISFYING THE SCALAR HELMHOLTZ EQUATION $(\nabla^2 + \kappa_2^2) \chi = 0$. EQUATION (A) IN COMPONENT FORM IS,

$$\psi_r = 0, \quad \psi_\theta = 0, \quad \psi_z = \kappa_2^2 \chi \quad (B)$$

TO THE SCALAR FUNCTION χ . SUBSTITUTING SOLUTION (A) INTO $\vec{u} = \vec{\nabla} \phi + \vec{\nabla} \times \vec{\psi}$ YIELDS $\vec{u} = \vec{\nabla} \phi + \kappa_2^2 (\vec{\nabla} \times \chi) \times \vec{e}_\theta$, WHICH ARE THE DISPLACEMENT-DEPENDENT POTENTIAL RELATIONS FOUND IN EQS. (4) ABOVE.

WE REPEAT THAT IN THE ABSENCE OF VISCOSITY (i.e., $M=N=0$, i.e., ELASTICITY), SOLUTION (A) IS THEN THE SOLENOIDAL SOLUTION OF THE VECTOR HELMHOLTZ EQUATION WITH A REAL RATHER THAN COMPLEX PROPAGATION CONSTANT, PROVIDED THAT χ SATISFIES A SCALAR HELMHOLTZ EQ. WITH THE SAME REAL PROPAGATION CONSTANT $\lambda_2 = \omega/c_s$. HENCE, DAMPING DUE TO VISCOSITY IS ACCOUNTED FOR IN THESE MODELS BY MEANS OF COMPLEX WAVENUMBERS κ_1^2 AND κ_2^2 RELATED TO THE MATERIAL PROPERTIES AS SHOWN ABOVE IN (9). κ_1 & κ_2 ARE ASSOCIATED WITH LONGITUDINAL AND TRANSVERSE WAVES RESPECTIVELY. NOTE THAT IN THE MAXWELL MODEL THE PREVIOUS STATEMENT IS TRUE ONLY WHEN $C_1^2 \ll C_2^2$. OTHERWISE κ_1 "CONTAINS" BOTH TYPES OF WAVES.

AR POTENTIALS WHICH SATISFY THE HELMHOLTZ'S EQS. WITH COMPLEX κ 'S IN (10), THEN THE FIELD EQS. (7) ARE AUTOMATICALLY (MS OF THE POTENTIALS BY EQS. (4) & (8) RESPECTIVELY. THIS TECHNIQUE, WHICH IS BASIC TO SET-UP AND SOLVE BOUNDARY VALUE CITY, AND THAT IS WHY THIS CHART WAS DEVELOPPED FOR ALL THESE CASES AND VISCOELASTIC MODELS. THE READER FAMILIAR ROMAGNETIC THEORY IN A COULOMB ($\vec{\nabla} \cdot \vec{\psi} = 0$) RATHER THAN A LORENTZ (i.e., $\vec{\nabla} \cdot \vec{\psi} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$) GAUGE. ABSENCE OF VISCOSITY, (i.e., $\lambda_e = \mu_e = 0$ FOR THE KELVIN-VOIGT MODEL AND/OR $\alpha = \beta = 0$ FOR THE MAXWELL MODEL.) E) ABOVE BY THE STANDARD TRICK OF REPLACING λ_e IN THE PLANE-STRAIN CASE BY λ_e DEFINED TO BE: $\lambda_e = 2\lambda_e \mu_e / (\lambda_e + 2\mu_e)$. M D) & E) BY SETTING $u_\theta = 0$ AND $\frac{\partial}{\partial \theta}$ [ANY VARIABLE] = 0. THEN THERE IS ONLY r -DEPENDENCE, (i.e., $u_r = u_r(r)$ & $\phi = \phi(r)$ & $u_\theta = 0$ or $\partial \phi / \partial r$). SO, THIS AMOUNTS TO HAVING $3\lambda_e + 2\mu_e = 0$ (KELVIN-VOIGT) OR $3\alpha + 2\beta = 0$ (MAXWELL), SO THE FORMULAS THEN SIMPLIFY A LITTLE.

$2\epsilon_{\theta\theta} = \frac{1}{r^2} \left[\frac{\partial^2 \psi}{\partial \theta^2} - 2\frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} - 2\frac{\partial \psi}{\partial r} \frac{\partial \psi}{\partial \theta} + \frac{\partial^2 \psi}{\partial \theta^2} - 2\frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right] - \kappa_2^2 \left[\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right]$ $\epsilon_{\theta\theta} = \frac{1}{r^2} \left[\frac{\partial^2 \psi}{\partial \theta^2} - 2\frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right] - \kappa_2^2 \left[\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right]$ <p>WE WILL SEE BELOW THAT IT IS NOT CONVENIENT TO WRITE ψ IN TERMS OF ϕ & ψ_1 SINCE ψ_1 SATISFIES</p>	$2\epsilon_{\theta\theta} = 2 \frac{\partial^2 \psi}{\partial \theta^2} \left[\frac{1}{r^2} \frac{\partial \psi}{\partial \theta} (\phi + \frac{\partial \psi}{\partial r} (r\psi)) \right] + \kappa_2^2 \frac{\partial^2 \psi}{\partial \theta^2}$ <p>WE WILL SEE BELOW THAT IT IS NOT CONVENIENT TO WRITE ψ IN TERMS OF ϕ & ψ_1 SINCE ψ_1 SATISFIES</p>
<p>7) FIELD EQUATIONS:</p> $\nabla^2 \psi = \frac{1}{r^2} \left[\frac{\partial^2 \psi}{\partial \theta^2} - 2\frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right] + \left[1 + \frac{1}{r^2} \right] \left[\frac{\partial^2 \psi}{\partial \theta^2} - 2\frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right] + \left[1 + \frac{1}{r^2} \right] \left[\frac{\partial^2 \psi}{\partial \theta^2} - 2\frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right]$ <p>WHERE, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$</p> $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}$ $\kappa_2^2 = \rho \omega^2 / (\mu_e - i\omega \mu_v)$ <p>FORM (2) IS NOT AS USEFUL.</p>	<p>8) STRESS-INDEPENDENT POTENTIAL RELATIONS:</p> $\epsilon_{rr} = -[\lambda_e - i\omega \lambda_v] \kappa_1^2 \phi + 2(\mu_e - i\omega \mu_v) \left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right] + \kappa_2^2 \left[\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right]$ $\epsilon_{\theta\theta} = -[\lambda_e - i\omega \lambda_v] \kappa_1^2 \phi + 2(\mu_e - i\omega \mu_v) \left[\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right] + \kappa_2^2 \left[\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right]$ $\epsilon_{r\theta} = -[\lambda_e - i\omega \lambda_v] \kappa_1^2 \phi + 2(\mu_e - i\omega \mu_v) \left[\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right] + \kappa_2^2 \left[\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right]$ $\epsilon_{\theta\theta} = (\mu_e - i\omega \mu_v) \left[\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right] + \kappa_2^2 \left[\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right]$ $\epsilon_{r\theta} = (\mu_e - i\omega \mu_v) \left[\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right] + \kappa_2^2 \left[\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \right]$ <p>THESE ARE EQS. (6) SUBSTITUTED INTO (5). IF, $\vec{u} = \nabla \phi + \nabla \times \vec{\psi}$ THEN, $\Delta = \nabla^2 \phi = -\kappa_1^2 \phi$.</p>
<p>9) HELMHOLTZ'S EQS. FOR THE SCALAR AND VECTOR POTENTIALS ϕ AND $\vec{\psi}$:</p> $\nabla^2 \phi + \kappa_1^2 \phi = 0 \quad \nabla^2 \vec{\psi} + \kappa_2^2 \vec{\psi} = 0 \quad \nabla \cdot \vec{\psi} = 0$ <p>WHERE, $\kappa_1^2 = \frac{\omega^2}{c^2(1-i\omega M)}$, $\kappa_2^2 = \frac{\omega^2}{c^2(1-i\omega N)}$, $M = \frac{\lambda_e + 2\mu_e}{\lambda_e + 2\mu_e}$, $N = \frac{\mu_e}{\mu_e}$, $c^2 = \frac{\lambda_e + 2\mu_e}{\rho}$, $c^2 = \frac{\mu_e}{\rho}$.</p> <p>THESE ARE HELMHOLTZ EQS. WITH COMPLEX PROPAGATION CONSTANTS. THE VECTOR EQ. IS FURTHER SUBJECTED TO THE SOLENOIDAL (GAUGE) CONDITION SHOWN.</p>	<p>9) HELMHOLTZ'S EQS. FOR THE SCALAR AND VECTOR POTENTIALS ϕ AND $\vec{\psi}$:</p> $\nabla^2 \phi + \kappa_1^2 \phi = 0 \quad \nabla^2 \vec{\psi} + \kappa_2^2 \vec{\psi} = 0 \quad \nabla \cdot \vec{\psi} = 0$ <p>WHERE, $\kappa_1^2 = \frac{\omega^2}{c^2(1-i\omega M)}$, $\kappa_2^2 = \frac{\omega^2}{c^2(1-i\omega N)}$, $M = \frac{\lambda_e + 2\mu_e}{\lambda_e + 2\mu_e}$, $N = \frac{\mu_e}{\mu_e}$, $c^2 = \frac{\lambda_e + 2\mu_e}{\rho}$, $c^2 = \frac{\mu_e}{\rho}$.</p> <p>AND, $c^2 = (\lambda_e + 2\mu_e)/\rho$, $c^2 = \mu_e/\rho$.</p> <p>THESE ARE HELMHOLTZ EQS. WITH COMPLEX PROPAGATION CONSTANTS. THE VECTOR EQ. IS FURTHER SUBJECTED TO THE SOLENOIDAL (GAUGE) CONDITION SHOWN.</p>
<p>10) HELMHOLTZ'S EQS. FOR THE INDEPENDENT SCALAR POTENTIALS:</p> $\nabla^2 \phi + \kappa_1^2 \phi = 0 \quad \nabla^2 \psi + \kappa_2^2 \psi = 0 \quad \nabla^2 \chi(r, \theta, \phi) = 0$ <p>WHERE THE COMPLEX κ_1^2 & κ_2^2 ARE AS GIVEN IN 9) AND ∇^2 AS GIVEN IN 7).</p>	<p>10) HELMHOLTZ'S EQS. FOR THE INDEPENDENT SCALAR POTENTIALS:</p> $\nabla^2 \phi + \kappa_1^2 \phi = 0 \quad \nabla^2 \psi + \kappa_2^2 \psi = 0 \quad \nabla^2 \chi(r, \theta, \phi) = 0$ <p>WHERE κ_1^2 & κ_2^2 ARE THE COMPLEX QUANTITIES. IF WE SET $\psi = -\frac{\partial \psi}{\partial r}$ IN THE LAST EQ. THIS IS WHY IT IS MORE CONVENIENT TO WRITE IN TERMS OF ψ RATHER THAN ψ_1.</p>
<p>11) SOLENOIDAL SOLUTION OF THE VECTOR HELMHOLTZ EQ. IN GENERAL SPHERICAL COORDINATES:</p> <p>THE SOLENOIDAL SOLUTION OF $(\nabla^2 + \kappa_2^2) \vec{\psi} = 0$ IS THE SOLUTION OF $\nabla \times (\nabla \times \vec{\psi}) = \kappa_2^2 \vec{\psi}$ AND IT IS,</p> $\vec{\psi} = \text{curl } \vec{F} \psi + \text{curl } \text{curl } \vec{F} \chi = (\nabla \psi) \times \vec{r} + \vec{r} \times \nabla \psi + \kappa_2^2 \vec{F} \chi + \nabla (\nabla \cdot \vec{F} \chi) - \vec{F} \nabla^2 \chi$ <p>PROVIDED $\psi(r, \theta, \phi)$ & $\chi(r, \theta, \phi)$ SATISFY SCALAR HELMHOLTZ EQS. $(\nabla^2 + \kappa_2^2) \psi = 0$, $(\nabla^2 + \kappa_2^2) \chi = 0$. NOTE THAT ψ IS NOT $(\vec{\psi})$. OPENING SOLUTION (A) INTO ITS COMPONENTS WE FIND,</p> $\psi_r = \frac{1}{r^2} \frac{\partial \chi}{\partial r} + \kappa_2^2 \chi = -\frac{1}{r^2} \frac{\partial \chi}{\partial r} - \frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} - \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \chi}{\partial \phi^2}$ $\psi_\theta = -\frac{1}{r^2} \frac{\partial \chi}{\partial \theta} + \frac{1}{r^2} \frac{\partial \chi}{\partial \theta} + \frac{\partial^2 \chi}{\partial \theta^2}$ $\psi_\phi = -\frac{1}{r^2} \frac{\partial \chi}{\partial \phi} + \frac{1}{r^2} \frac{\partial \chi}{\partial \phi} + \frac{\partial^2 \chi}{\partial \phi^2}$ <p>WHICH RELATE $\psi_r, \psi_\theta, \psi_\phi$ TO ψ AND χ.</p> <p>USING THE AUXILIARY RELATIONS $\nabla \times (\nabla \times \vec{\psi}) = \nabla (\nabla \cdot \vec{\psi}) - \nabla^2 \vec{\psi}$ AND $\nabla \times (\vec{F} \chi) = (\nabla \chi) \times \vec{r}$ WE CAN NOW SUBSTITUTE SOLUTION (A) INTO $\vec{u} = \nabla \phi + \nabla \times \vec{\psi}$ TO OBTAIN $\vec{u} = \nabla \left[\phi + \frac{\partial \chi}{\partial r} (r\psi) \right] + \kappa_2^2 \left[\vec{F} \psi + (\nabla \psi) \times \vec{r} \right]$ AS WE HAD IN EQS. 4. IN THE ABSENCE OF VISCOSITY (i.e., $M=N=0$, i.e., ELASTICITY) EQ. (A) ABOVE IS THE SOLENOIDAL SOLUTION OF THE VECTOR HELMHOLTZ EQ. NOW WITH A REAL RATHER THAN COMPLEX PROPAGATION CONSTANT $\kappa_2^2 = \omega^2/c^2$ PROVIDED ψ & χ SATISFY SCALAR HELMHOLTZ EQS. WITH THE SAME REAL PROPAGATION CONSTANT κ_2^2.</p>	<p>11) SOLENOIDAL SOLUTION OF VECTOR HELMHOLTZ EQ. IN GENERAL SPHERICAL COORDINATES:</p> <p>THE SOLENOIDAL SOLUTION OF $(\nabla^2 + \kappa_2^2) \vec{\psi} = 0$ IS THE SOLUTION OF $\nabla \times (\nabla \times \vec{\psi}) = \kappa_2^2 \vec{\psi}$ AND IT IS,</p> $\vec{\psi} = (\nabla \psi) \times \vec{r} + \vec{r} \times \nabla \psi + \kappa_2^2 \vec{F} \chi + \nabla (\nabla \cdot \vec{F} \chi) - \vec{F} \nabla^2 \chi$ <p>OR IN COMPONENT FORM, $\psi_r = 0$, $\psi_\theta = 0$, $\psi_\phi = 0$. (NOTE THAT ψ IS NOT $(\vec{\psi})$). PROVIDED THAT SUBSTITUTING SOLUTION (A) INTO $\vec{u} = \nabla \phi + \nabla \times \vec{\psi}$ AS WE HAD A SOLUTION (A) HOLDS IN THE ABSENCE OF ELASTICITY. IT IS THEN THE SOLUTION OF EQ. WITH A REAL PROPAGATION CONSTANT. SATISFIES THE SCALAR HELMHOLTZ EQ. WITH $\kappa_2^2 = \kappa_2^2 - 1/r^2$ SINCE TO SOLVE THAN (10) (2) BUT IT IS STILL IS POSSIBLE FOR κ_2^2 AS GENERAL AS, $\kappa_2^2 = \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \chi}{\partial \phi^2}$ WHERE f, g, h, ARE CONSTANT FUNCTIONS OF THE INDICATED</p>
<p>12) REMARKS:</p> <p>(1) REMARKS a), b), c) OF THE CYLINDRICAL CASES TO THE LEFT, ALSO HOLD HERE.</p> <p>(2) NOTE AGAIN THAT IN SPITE OF THE APPARENT GENERALITY OF THE SOLENOIDAL SOLUTION (1) (A) ABOVE WITH ITS VECTOR OPERATORS, IT IS ONLY VALID IN SPHERICAL COORDINATES. NOTE FURTHER THAT THE TWO PARTS (i.e., $\text{curl } \vec{F} \psi$ AND $\text{curl } \text{curl } \vec{F} \chi$) ARE DIFFERENT EVEN WHEN $\psi = \chi$, ACTUALLY, THEY ARE PERPENDICULAR WHEN $\psi = \chi$.</p> <p>(3) THERE IS NO WAY TO HAVE A PLANE-STRAIN OR PLANE-STRESS SITUATION IN SPHERICAL COORDS. THE CLOSEST WE CAN GET TO THEM IS THE AXIALLY-SYMMETRIC CASE (3) TO THE RIGHT.</p>	<p>12) REMARKS:</p> <p>(1) REMARKS a), b) & c) OF THE CYLINDRICAL CASES TO THE LEFT, ALSO HOLD HERE. REMARK a) IS THE MAXWELL MODEL IN SPHERICAL COORDS. AS AN EXERCISE TO THE READER.</p> <p>(2) THIS ENTIRE CHART, WHICH IS VALID FOR THE MAXWELL MODEL, IS A PARTICULAR COMPANION "TABLE 1" VALID FOR ARBITRARY (G. GAUNAUD, 1975).</p>

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in a more or less complicated fashion depending on the model being used. For example, the relation is more complicated for the Maxwell than for the Kelvin model, in any coordinate system. These tables provide us with the methodology and information needed to solve boundary-value-problems of viscoelasto-dynamics for spherical or cylindrical geometries, in the Kelvin-Voigt or Maxwell models. The approach and material covered in these tables seems to be novel as applied to viscoelasticity. The reader familiar with electro-magnetic theory will quickly notice the analogy of the presentation in this table to that of electrodynamic theory in the less familiar Coulomb ($\vec{\nabla} \cdot \vec{\psi} = 0$), rather than Lorentz (ie, $\vec{\nabla} \cdot \vec{\psi} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$) gauge.

To verify that all the entries in the tables check with each other, we will illustrate the use of the formulas by analyzing one example. We arbitrarily select column D, which deals with the Kelvin-Voigt model in cylindrical coordinates for the subcase of plane-strain.

1. Definition of Plane-Strain

In plane-strain, the strain tensor is two-dimensional and all strain components with a subindex z (axial coordinate) vanish. The stress-tensor is not, and there is a nonvanishing normal stress τ_{zz} . Another equivalent way to define this situation is by stating that the axial displacement u_z vanishes, and that the other two displacement components u_r and u_θ are z -independent. Plane-strain is a particularly useful approximation for bodies which are very long in the z -direction.

2. Constitutive Relations

These are the relations between the stress and strain field components. There are three normal and one shear-stress components related to two normal and one shear-strain components. Stress and strain are independent concepts. Only when these two notions are linked through constitutive relations, is that a visco-elastic theory is formed. The first of these four relations is,

$$\tau_{rr} = (\lambda_e - i\omega\lambda_v) \Delta + 2(\mu_e - i\omega\mu_v) \epsilon_{rr}$$

where Δ is the trace of the strain tensor (ie, $\epsilon_{rr} + \epsilon_{\theta\theta}$) and it is called the dilatation. The other three relations are shown in Table 2.

3. Strain-Displacement Relations

Since the elastic or viscous constants do not appear in these relations, they are the same as in elasticity, namely,

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad 2\epsilon_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

all other strain-components vanishing by definition of plane-strain.

4. Displacement - Independent Potential Relation

These relations are intimately linked to the solenoidal solution of the vector Helmholtz equation in plane polar coordinates which is discussed below in item (11). The relations can be written as a single vector equation as follows,

$$\vec{u} = \vec{\nabla}\phi + (\vec{\nabla}\psi_z) \times \hat{e}_z$$

where ϕ and ψ_z are the two independent scalar potentials needed in this plane-strain formulation. It is shown in item (11) that $\psi_z = \kappa_2^2 \chi$, where χ is a solution of the scalar Helmholtz's equation $(\nabla^2 + \kappa_2^2) \chi = 0$.

5. Stress - Displacement Relations

If the strain-displacement relations (3) are substituted into the constitutive relations (2), the result is the stress-displacement relations. The first such relation is,

$$\tau_{rr} = [\lambda_e - i\omega\lambda_v] \Delta + 2 [\mu_e - i\omega\mu_v] \frac{\partial u_r}{\partial r}$$

where the dilatation Δ is now expressed in terms of the displacement components as follows,

$$\Delta = \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

6. Strain - Independent Potential Relations

If the displacement-independent potential relations (4) are substituted into the strain-displacement relations (3), the result is the strain-potential relation. The first of these three is,

$$\epsilon_{rr} = \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi_z}{\partial \theta} \right)$$

and analogously for the other two. Here ϕ and ψ_z are the two independent scalar potentials. Note that ψ_z is the axial component of the vector potential $\vec{\psi}$. Further, $\psi_z = \kappa_2^2 \chi = -\nabla^2 \chi$ where χ is another independent scalar potential which can also be used here. (See item (11) below)

7. Field Equations

These are the Navier equations given initially in (i). For harmonic time-dependence they can be written as follows,

$$\nabla^2 \vec{u} + \left[1 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v} \right] \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) + \kappa_2^2 \vec{u} = 0$$

where $\kappa_2^2 = \rho\omega^2 [\mu_e - i\omega\mu_v]^{-1}$, and $\Delta = \text{dilatation} = \vec{\nabla} \cdot \vec{u}$. For plane-strain in cylindricals, the quantities $\nabla^2 \vec{u}$ and $\vec{\nabla} \Delta$ take on the following simplified forms,

$$\nabla^2 \vec{u} = \left\{ \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right\} \hat{e}_r + \left\{ \nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right\} \hat{e}_\theta$$

$$\text{grad } \Delta = \frac{\partial \Delta}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \Delta}{\partial \theta} \hat{e}_\theta$$

and also we have,

$$\vec{u} = u_r \hat{e}_r + u_\theta \hat{e}_\theta$$

$$\text{and, } \nabla^2 u_r = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) u_r(r, \theta).$$

8. Stress - Independent Potential Relations

If the strain-potential relations (6) are substituted into the constitutive relations (2), we obtain the stress-potential relations. One form of the first of these relations is,

$$\tau_{rr} = [\lambda_e - i\omega\lambda_v] \Delta + 2 [\mu_e - i\omega\mu_v] \left\{ \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial \psi_z}{\partial \theta} \right] \right\}$$

$$\text{where, } \Delta = \vec{\nabla} \cdot \vec{u} = \vec{\nabla} \cdot [\vec{\nabla} \phi + \vec{\nabla} \times \vec{\psi}] = \nabla^2 \phi = -\kappa_1^2 \phi.$$

The second form can be found by setting, $\psi_z = \kappa_2^2 \chi$. The formulas for the other three stresses present in this case can be found in Table 2.

9. Helmholtz's Equations For The Scalar And Vector Potentials

Substituting $\vec{u} = \text{grad } \phi + \text{curl } \vec{\psi}$ into the Navier equations written in the alternative form,

$$(\lambda^* + 2\mu^*) \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) - \mu^* \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) + \mu^* \kappa_2^2 \vec{u} = 0$$

where $\lambda^* = \lambda_e - i\omega\lambda_v$, $\mu^* = \mu_e - i\omega\mu_v$, and $\mu^* \kappa_2^2 = \rho\omega^2$ we eventually find,

$$(\lambda^* + 2\mu^*) \vec{\nabla} \{ \nabla^2 \phi + \kappa_1^2 \phi \} + \mu^* \vec{\nabla} \times \{ \nabla^2 \vec{\psi} - \vec{\nabla} (\vec{\nabla} \cdot \vec{\psi}) + \kappa_2^2 \vec{\psi} \} = 0 \quad \text{where}$$

$\kappa_1^2 = \rho \omega^2 [\lambda_e + 2\mu_e - i\omega (\lambda_v + 2\mu_v)]^{-1}$ and κ_2^2 is as we defined it above in item (7). It is obvious that the above expression is satisfied if,

$$\nabla^2 \phi + \kappa_1^2 \phi = 0 \quad \text{and} \quad \nabla^2 \vec{\psi} + \kappa_2^2 \vec{\psi} = 0 \quad \text{provided that} \\ \vec{\nabla} \cdot \vec{\psi} = 0 \quad (\text{ie, "Coulomb" gauge}).$$

10. Helmholtz's Equations For The Independent Scalar Potentials

Since in this case, $\vec{\psi} = \hat{e}_z \psi_z$ then, $\nabla^2 \vec{\psi} = \hat{e}_z \nabla^2 \psi_z$, and the result is,

$$(\nabla^2 + \kappa_1^2) \phi = 0, \quad (\nabla^2 + \kappa_2^2) \psi_z = 0$$

11. Solenoidal Solution of The Vector Helmholtz Equation

It is not hard to show that the solenoidal solution of $(\nabla^2 + \kappa_2^2) \vec{\psi} = 0$

is given by, $\vec{\psi} = \vec{\nabla} \times [\vec{\nabla} \times (\hat{e}_z \chi)] = \hat{e}_z \kappa_2^2 \chi$ where $\chi(r, \theta)$ is a scalar function satisfying the scalar Helmholtz's equation $(\nabla^2 + \kappa_2^2) \chi = 0$. In component form, the above relation is $\psi_r = 0$, $\psi_\theta = 0$, and $\psi_z = \kappa_2^2 \chi$. These are the relations between the three cylindrical components of the vector potential $\vec{\psi}$ and the scalar function $\chi(r, \theta)$. Note that, indeed, $\vec{\nabla} \cdot \vec{\psi} = 0$. Note that in the absence of viscosity (ie, $\lambda_v = 0$, $\mu_v = 0$) the complex propagation constants κ_1 and κ_2 become real. Thus viscous damping is accounted for in these models by complex propagation constants, related to the material "constants" as shown above in items (7) and (9).

12. Remarks

a) In cylindrical coordinates the plane-stress results are derivable from the plane-strain results by replacing the elastic constant λ in the plane-strain results, by the fictitious "constant" λ_e defined to be, $\lambda_e = 2\lambda\mu_e / (\lambda_e + 2\mu_e)$. In spherical coordinates there is no way to define the plane-stress subcase. In plane-stress, the stress tensor is two-dimensional, which means that all stress-components with a subindex z, vanish. The strain-tensor is not two-dimensional and there is an ϵ_{zz} nonvanishing strain. This case is ideally suited for bodies which are very thin in one direction, the z-direction.

b) Plane-strain (or plane-stress) with axial-symmetry about the z-direction can be obtained from the results in the eleven items above by merely setting $u_\theta = 0$ and $\frac{\partial}{\partial \theta}$ [any variable] = 0. There is only radial dependence in this situation.

c) All the above applies to the Kelvin-Voigt model in cylindrical (actually, plane polar) coordinates, for the plane-strain subcase. This is Column D of Table 2. There are twelve cases covered in Tables 1 and 2 and this one was intended as an example to show how the various entries are derived from the others, how they check with each other, and how one could proceed to drive any other case. We have used the information in these tables to rigorously set up and solve acoustic scattering problems where the scattering bodies are elastic cylinders and spheres with their outer surface coated with layers of viscoelastically absorbing materials.⁶ One reason to have written this report is to gather these fundamental viscoelasticity relations, which are not available elsewhere to this degree of detail and approach, in one document that we could refer to in future work as the place where the theoretical background is derived and presented, in the form in which we will use it.

d) Some authors have stated that in order to solve problems involving absorptive bodies one must deal with Helmholtz's equation with a complex propagation constant K . This is indeed correct but there is much more to it than just that. Just with a complex K we would not know how the real and imaginary parts of K are related to the elastic and viscous material constants of the solid in the various viscoelastic models that one could use. In fact, we would not know this relationship in any model. Furthermore, we would not know how to relate the stresses and displacements in the body to the solution of that Helmholtz equation with a complex K . Hence, although the idea is correct, in practice one really needs to derive all the detailed information contained in these tables, and that is why we developed them. Careful examination of these tables shows that the problem is really harder than anticipated, since we must solve not one but several (ie, three) Helmholtz's equations with various (ie, two) complex propagation constants which are different, and then go through various other sets of equations (ie, 4 and 8) to obtain the displacements and stresses from the solutions of those Helmholtz's equations. We finally point out that this procedure yields different results in each one of the various models and cases presented there.

e) Column A, Table 1 shows some general formulas valid for all coordinate systems. Note, however, that since the solution of the vector telegraph equation varies with the coordinate system, not many general entries can be filled in that column.

⁶ G. C. Gaunard, "Sound Scattering from an Elastic Cylinder Covered With a Viscoelastic Coating", JASA 58, S101, 1975. Also, Proc. of XII, U.S.-Fed. Republic of Germany, Hydroacoustics Symposium, Munich, Germany, Vol 1, Part II, pp 4-31, May 1975 (U)

or also, $\phi(x,t) = A e^{\pm \kappa x + i\omega t}$ where, $\kappa^2 = -\frac{\omega^2}{C_p^2 (1 + i\omega P)}$ (3)

Here C_p is real and the \pm signs in front of κx describe waves travelling to the right or left of some origin. (If one were interested in time-dependence $\exp(-i\omega t)$ we would replace i by $-i$ in all these results.)

Now let us consider the standard one-dimensional wave equation,

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2} \quad (4)$$

The solution of eq. (4) for harmonic time-dependence $\exp(i\omega t)$ can analogously be written as,

$$\phi(x,t) = A e^{i[\pm \kappa x + \omega t]} \quad \text{where } \kappa^2 = \frac{\omega^2}{C^2} \quad (5)$$

or also, $\phi(x,t) = A e^{\pm \kappa x + i\omega t}$ where $\kappa^2 = \frac{\omega^2}{C^2}$.

It is obvious that solutions (2) and (5) and also (3) and (6) can be made equal by setting $C^2 = C_p^2 (1 + i\omega P)$. It is clear that we can solve the telegraph equation with a real propagation speed C_p by ignoring the damping term, i.e., by solving the wave-equation with a suitably chosen complex propagation speed. Complex speeds are artificially produced by fictitious complex elastic constants that one can introduce for this purpose.

Let us now look at the case of shear waves in a solid. Define a complex shear modulus $\mu^* = \mu' + i\mu'' = \mu' (1 + i\delta)$, where $\delta = \mu''/\mu'$.

The equations of linear viscoelasticity are the same of those of elasticity (i.e., no viscosity) if λ_e and μ_e in the elasticity equations are replaced by the

quantities, $\lambda^* = \lambda_e + i\omega\lambda_v$ $\mu^* = \mu_e + i\omega\mu_v$ (7)

NOTE: This is true only in the Kelvin-Voigt viscoelastic model with assumed time-dependence of the form $\exp(i\omega t)$.

Clearly $\mu_e = \mu'$ and $\mu_v = \mu''/\omega$. Thus, for shear waves,

$$P = N = \frac{\mu_v}{\mu_e} = \frac{\mu''}{\omega\mu'} = \frac{\delta}{\omega} \quad \text{and} \quad C = C_s \sqrt{1 + i\omega N} = \sqrt{\frac{\mu' + i\mu''}{\rho}} = \sqrt{\frac{\mu^*}{\rho}} \quad (8)$$

Hence, for harmonic shear waves in the Kelvin model, by solving the wave-equation, with a complex shear modulus, we have the solution of the telegraph equation. This can also be done in the Kelvin model for longitudinal (or dilatational) waves.

It turns out that the Kelvin model with harmonic time-dependence assumed, is the only case where the viscosity coefficients λ_v, μ_v of the viscoelastic solid are linearly proportional to the imaginary parts λ''', μ''' of the "complex elastic constants" λ^*, μ^* , the proportionality factors being $1/\omega$. It is also the only case where the elastic constants λ_e, μ_e of the solid are just the real parts λ', μ' of the "complex elastic constants".

To show this, let us now consider the Maxwell model, where the springs and dashpots at each material point are now connected in series. It can be quickly shown that the field equations and constitutive relations of the Maxwell solid are the same as those of elasticity, provided that the elastic constants λ_e, μ_e of the elasticity equations are replaced by the operators,

$$\lambda_e \rightarrow \frac{\frac{\partial}{\partial t}}{2\beta + \frac{\partial}{\partial t}} \left\{ \frac{\lambda_e (2\beta + \frac{\partial}{\partial t}) - 2\alpha\mu_e}{3\alpha + 2\beta + \frac{\partial}{\partial t}} \right\}, \quad \mu_e \rightarrow \frac{\mu_e \frac{\partial}{\partial t}}{2\beta + \frac{\partial}{\partial t}} \quad (9)$$

where α, β are the viscosity coefficients of the Maxwell model (which are analogous to λ_v, μ_v of the Kelvin model). For time-dependence of the form $\exp(-i\omega t)$ we can call these operators by the name "complex elastic constants" λ^*, μ^* ie,

$$\lambda^* = \frac{i\omega}{i\omega - 2\beta} \left\{ \frac{\lambda_e (2\beta - i\omega) - 2\alpha\mu_e}{3\alpha + 2\beta - i\omega} \right\}, \quad \mu^* = \frac{\mu_e i\omega}{i\omega - 2\beta} \quad (10)$$

It is obvious that if there is no viscosity (ie, $\alpha = 0, \beta = 0$) these quantities λ^*, μ^* would reduce to λ_e, μ_e respectively. It is also clear that α and β are now not proportional to the imaginary parts of λ^* and μ^* respectively. Furthermore, λ_e and μ_e are not the real parts of λ^* and μ^* anymore. Thus the "trick" of the complex elastic constants does not work here at all. The "trick" also fails for the standard viscoelastic model, or any more complicated model which contains at least one Maxwell element. In fact it fails even when there is no Maxwell element in the model provided there is more than one Kelvin element. Thus, only for one single Kelvin element will it "work."

For the reasons given above we believe that the most clear, systematic and natural way to handle viscoelastic problems of this sort is to measure the (real) elastic constants of the material independently from the (real) viscous coefficients and then use those numerical values in the field-equations when analytically solving them. Of course, we must keep the damping terms in the field equations. The final result will be complex, but this fact is due to the presence of the damping terms rather than because of any "complex moduli" we want to fictitiously introduce because of our insistence on ignoring the damping terms instead. We have seen that the complex moduli "trick" becomes exceedingly difficult, in fact impossible, in any case other than the Kelvin-Voigt model with harmonic time-dependence.

1. The Correspondence Theorem

A very useful theorem that permits us to relate solutions of elastic problems to those of viscoelasticity (in any model) is the Correspondence Theorem.⁷ This theorem has a conceptually very simple statement, but it is very difficult to apply it to actual cases. The theorem basically says that if we want the solution to a viscoelasticity problem (static or dynamic, and in any model) we should first solve the "corresponding" elastic problem without viscosity. Then we Laplace-transform the solution. Then we replace the elastic constants appearing in that Laplace-transformed solution, by certain "memory functions" (which vary with the viscoelastic model being used). The result of this replacement is the Laplace transform of the solution of the "corresponding" viscoelastic problem. Inverting it, we have the solution of the viscoelastic problem we wanted to solve. We here recall the Laplace Transform pair,

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt, \quad f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s) e^{st} ds.$$

The prescription given by the Correspondence Theorem is very straight forward. It turns out in practice that when the elastic constants are replaced by those "memory functions", the resulting Laplace-transformed solution that must now be inverted is quite formidable in most cases of interest (ie, dynamic cases). We should point out that the memory functions depend on the Laplace transform variable s in a more or less complicated manner depending on whatever viscoelastic model one uses. See equations (7) or (10) with $i\omega$ replaced by s , for the Kelvin or the Maxwell model respectively.

I don't really want to discuss the Correspondence Theorem or its applications. My point is that those "memory functions" mentioned in it, are precisely the equivalent or the analogue of those "complex-elastic constants" that some authors try to introduce fictitiously in real-space rather than in the Laplace-space, before inversion to the real time-domain. Hence, the way to use those "complex elastic constants" in a way that "works," is really in the light of the correspondence theorem. Unfortunately, that is quite a difficult task.

We have already pointed out that another method which "works" is to measure the (real) elastic and viscous constants of the material, each set independently of the other, and then solve the field-equations with the damping terms included, with those numerical values measured for the constants. This method is also general and works for any viscoelastic model and it is, incidentally, the way viscous flow problems are attacked in Fluid Mechanics.

⁷ See A. Cemal Eringen, "Continuum Mechanics," John Wiley and Sons, Inc., 1967, Article 9.12, p. 368.

Observation of Table 1 and 2 shows why the telegraph-type equations we have discussed here are so important in viscoelasto-dynamical problems. Note that the telegraph-type equations for the potentials ϕ , ψ , χ are of the type given here in eq. (1), for the Kelvin-Voigt model only. Note, for example, that for the Maxwell model they are substantially different. (Table 1, eqs. C-10). The equation for ϕ is,

$$\left[2\beta + 4\alpha \frac{c_s^2}{c_d^2} + \frac{\partial}{\partial t} \right] \nabla^2 \phi = \frac{1}{c_d^2} \left\{ \left[3\alpha + 4\beta + \frac{\partial}{\partial t} \right] \frac{\partial^2 \phi}{\partial t^2} + 2\beta (3\alpha + 2\beta) \frac{\partial \phi}{\partial t} \right\} \quad (11)$$

which is of the same "type" but not quite of the form given here in eq. (1).

We can conceptually always introduce complex moduli (as in eqs. (7) or (10) or their analogues for the "standard" model) provided we work in Laplace space as prescribed by the Correspondence Theorem. If we introduce them in the real time-domain and then try to physically identify the real (or imaginary) parts of these complex constants with the real elastic constants (or the viscous coefficients), then the process only "works" in the Kelvin-Voigt model. In this sense, "works" means that the equivalence of both approaches can be established.

One way to experimentally measure the absorption losses of material samples is the impedance tube or pulse tube. It is customary for the literature on this technique to report measurements of quantities such as E' and E'' or μ' and μ'' etc... This technique is a good source for the popularity of "complex moduli". We should note that the lossy samples tested in this fashion are always characterized by the Kelvin-Voigt viscoelastic model with harmonic time-dependence, an assumption that may not always be justified.

2. Conclusions

We summarize our points as follows,

(a) The continuum (infinite number of degrees-of-freedom) approach to viscoelasticity as a field-theory is the only way to go today. Discrete approaches leave much to be desired and are simplistically unrealistic.

(b) It looks like for some time to come, we will be dealing with the Kelvin-Voigt and Maxwell models, since any other model presents too many analytical difficulties.

(c) A systematic way to set up viscoelastic boundary-value-problems and solve the field equations in these two models, is presented here for cylinder and sphere problems. This approach is shown in Tables 1 and 2 which are self-explanatory.

(d) The most general method available to solve static or dynamic problems of viscoelasticity in any model is by means of the Correspondence Theorem. Here we must work in Laplace-space and the inversions are hard. This approach is totally equivalent (at least for the Kelvin and Maxwell models) to our approach described above in (3) and in Tables 1 and 2.

(e) Another technique, which holds for any model, is also given here. It consists of solving those resulting "telegraph-type" equations for the scalar potentials keeping in them the damping terms, but using real values for the elastic and viscous constants which are to be experimentally measured as it is done in Fluid Mechanics.

(f) We claim this technique is more systematic and less confusing than to ignore the damping terms introducing instead "complex elastic moduli". The complex moduli "trick" only works for the Kelvin model with harmonic time-dependence anyway.

(g) It seems clearer to us when talking about elastic constants, say, Young's modulus E , to think of the slope of the stress-strain curve as found in a tension test, than of that "complex Young's modulus" which contains the "loss" in its imaginary part, all because some want to solve wave equations rather than telegraph equations, particularly in the only situation when one is no harder to solve than the other.

(h) The determination of the solid's elastic constants should be kept separate and independent from the determination of the viscosity coefficients. I know this can be done in some instances, but I am not aware of how plausible this recommendation can be in all instances.

(i) Finally it is worth stating that the pulse tube measurements implicitly describe the viscoelastic "losses" in the sample by the Kelvin-Voigt model, an observation that escaped me (and perhaps others) until recently.

(j) Equation (10) can be used to express the complex shear and dilational moduli for the Maxwell model in terms of the viscosity coefficients and the elastic constants of the model as follows,

$$\begin{aligned} \mu^* &= \frac{\mu_e}{1 + \left(\frac{2\beta}{\omega}\right)^2} \left[1 - i \left(\frac{2\beta}{\omega}\right) \right] \quad \text{and} \\ \lambda^* + 2\mu^* &= \frac{1}{1 + \left(\frac{2\beta}{\omega}\right)^2} \cdot \frac{1}{1 + \left(\frac{3\alpha + 2\beta}{\omega}\right)^2} \left\{ \left[\lambda_e + 2\mu_e \right] \left[1 + \left(\frac{2\beta}{\omega}\right)^2 \right] + \right. \\ &+ \frac{4\alpha\mu_e}{\omega} \left(1 + \frac{2\beta}{\omega} \right) - i \left(\frac{3\alpha + 2\beta}{\omega} \right) \left[\lambda_e \left\{ 1 + \left(\frac{2\beta}{\omega}\right)^2 \right\} + \right. \\ &\left. \left. + 2\mu_e \left\{ \frac{4\beta}{\omega} + \frac{\alpha + \beta}{\omega} - \frac{\alpha + 2\beta}{\omega} \right\} \right] \right\}. \end{aligned}$$

These relations may be useful when trying to interpret "losses" in the light of the Maxwell model if one day one wishes, or the need arises to do so. It is obvious that these relations are considerably more complicated than the analogous ones for the Kelvin model, which are,

$$\mu^* = \mu_e [1 + i\omega\eta] \quad , \quad \lambda^* + 2\mu^* = (\lambda_e + 2\mu_e) [1 + i\omega\eta] \quad \text{where,}$$

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$$N = \frac{\mu_v}{\mu_e} \quad \text{and} \quad M = \frac{\lambda_v + 2\mu_v}{\lambda_e + 2\mu_e} .$$

(See Table 2, column A, item (9).)

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